

Equilibrium in Constrained Financial Markets*

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Abstract Redundant assets give rise to peculiar portfolios, called ‘link portfolios,’ under portfolio constraints. Link portfolios are jointly spanned by constrained null-income portfolios and form a linear subspace. The paper provides a general methodology for showing the existence of equilibrium under portfolio constraints by building two theoretical pillars to deal with link portfolios. The two pillars consist of the fundamental theorem of portfolio decomposition and the allocational equivalence between the original economy and the artificial economy built from projecting away link portfolios from the portfolio constraints. Investigating the existence of equilibrium in constrained financial markets boils down to finding a sufficient condition for the fundamental theorem of portfolio constraints to hold. The sufficient condition of the paper is general enough to encompass other sufficient conditions of the literature.

Keywords Incomplete Financial Markets, Portfolio Constraints, Financial Derivatives, Link Portfolios, Portfolio Decomposition, Equilibrium

JEL Classification G12, D52, C62, G11

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1. INTRODUCTION

Portfolio constraints occur naturally when asset trades are affected by institutional arrangements or informational asymmetry. In many cases, these constraints can be represented by a convex set in the portfolio space. For instance, Luttmer (1996) shows that a convex cone constraint is well-suited to describing short-selling restrictions, margin requirements, bid-ask spreads, and proportional transaction costs. Portfolio constraints are also useful in analyzing credit-based financial assets. Firms or individuals with poor credit may not qualify for loans that are accessible to entities that have high credit quality.¹ Information asymmetry about credit quality often leads to development of many customized or personalized financial contracts that cannot be publicly traded. If information about future states that is dispersed among agents is not fully revealed in equilibrium, they will trade assets with asymmetric information. Citanna and Villanacci (2000a,b) exploit the fact that the measurability restrictions imposed by informational asymmetry among agents can be represented by linear portfolio restrictions.

This paper provides a sufficient condition for the existence of equilibrium in financial markets under frictions such as short-selling restrictions, margin requirements, bid-ask spreads, and proportional transaction costs. We develop a modular approach to portfolio restrictions that consists of four steps based on Won and Hahn (2003, 2007). To analyze the behavior of financial derivatives in constrained financial markets, we introduce the concept of *link portfolio*. Link portfolios constitute a particular type of constrained null-income portfolios that generate a maximal linear subspace in the set of aggregate feasible portfolios.² The four-step approach of the paper is built on two theoretical pillars: the fundamental theorem of portfolio decomposition and the allocational equivalence between the original economy and the artificial economy built from projecting

¹Borrowing interest rates are significantly affected by the credit quality of the borrower. For instance, A-rated firms are not qualified to issue bonds in the bond markets for AAA-rated firms.

²Mathematically speaking, null-income portfolios are a portfolio in the kernel of the payoff matrix. Nontrivial null-income portfolios exist if the payoff matrix does not have full rank.

away link portfolios from the portfolio constraints.³ Investigating the existence of equilibrium in constrained financial markets boils down to finding sufficient conditions for the fundamental theorem of portfolio constraints to hold. We present a general sufficient condition on portfolio constraints which embraces portfolio restrictions of the literature such as Martins-da-Rocha and Triki (2005) and Aouani and Cornet (2009, 2011).

The modular approach of the paper to the existence of equilibrium in constrained financial markets consists of the following four steps. The first step provides the existence of equilibrium in economies that faces convex portfolio constraints without nonzero link portfolios. The second step is to formulate conditions for the fundamental theorem of portfolio decomposition to hold. In the third step, link portfolios are removed from the economy to create an artificial economy that belongs to the class of economies specified in the first step. The final step shows that the original and artificial economies share the same equilibrium allocations. This result guarantees the existence of equilibrium of the original economy.

The *fundamental theorem of portfolio decomposition*, the first pillar, states that each feasible portfolio can be split orthogonally into the income-determining portfolio and the link portfolio. Income-determining portfolios generate contingent income necessary to finance consumption in each state. Thus, what agents care about in choosing a portfolio in the budget set is the income-determining component of the portfolio. Income-determining portfolios have the nice property that they allow us to find equilibrium portfolios in a bounded set of portfolios. However, they alone need not be feasible under the portfolio constraints and moreover the set of income-determining portfolios may not be closed. By the conventional wisdom of the general equilibrium literature, the set of income-determining portfolios is required to be closed for the existence of equilibrium because they determine the magnitude of intertemporal income transfers. This need not be the case. In fact, the real hard problem with link portfolios arises when the individual sets of income-determining portfolios are not closed. The

³The origin of the two-pillar approach dates back to Won and Hahn (2007).

fundamental theorem of portfolio decomposition provides an answer about how to deal with the link portfolio problem. The first pillar provides a stepping stone for the second pillar, the allocational equivalence between equilibria of the original economy and the artificial economy where each agent faces the artificial portfolio constraint identified as the closure of income-determining components of the portfolios in the original portfolio constraints. Based on the fact that the artificial economy has equilibrium, the two pillars are combined to show the existence of equilibrium in the original economy. Specifically, the optimal portfolios of the original economy come from tracing back the optimal portfolios of the artificial economy to the original portfolio constraints through the portfolio decomposition theorem.

The fundamental theorem of portfolio decomposition is built on the *closedness condition for contingent income redistributions* (in short, CCIR condition) and two technical conditions (to be stated in Assumption (A7)) which account for the effect of link portfolios on the constrained set of risk-sharing opportunities. The CCIR condition exploits the intuition that a market-clearing portfolio allocation induces income redistribution among the agents in future contingencies. When redundant asset exists, the market-clearing portfolio allocation plus an allocation of null-income portfolios induces the same contingent income redistribution (CIR). Thus there may exist a set of CIR-inducing feasible portfolio allocations that need not clear the financial markets. For a given CIR, we choose a convergent sequence of aggregate portfolios each of which is the sum of a CIR-inducing feasible portfolio allocation over the agents. The CCIR condition holds if the limit aggregate portfolio is spanned by a feasible portfolio allocation that induces the same income redistribution. As illustrated in Example 5.1, the failure of the CCIR condition may preclude the existence of equilibrium. Assumption (A7) where the CCIR condition plays a pivotal role turns out to encompass all the distinct conditions on portfolio constraints studied in the literature.

Won and Hahn (2003) make a first attempt to develop a general methodology for portfolio decomposition, and illustrate how redundant assets contribute to risk sharing under portfolio constraints. The portfolio restrictions of the paper, Assumption (A7), cover portfolio constraints studied in the recent litera-

ture such as Martins-da-Rocha and Triki (2005) and Aouani and Cornet (2009, 2011). Aouani and Cornet (2011) impose a closedness condition on the set of individual incomes and their associated aggregate feasible portfolios. The portfolio restrictions of Martins-da-Rocha and Triki (2005) are so distinct from those of Aouani and Cornet (2011) that they do not imply each other. Both portfolio restrictions imply Assumption (A7) and thus, are sufficient for the fundamental theorem of portfolio decomposition to hold.

The effect of portfolio constraints on equilibrium was initially studied in Siconolfi (1989), which has inspired subsequent research on constrained financial markets. Balasko *et al.* (1990), and Benveniste and Ketterer (1992) assume that portfolio constraints are represented by linear homogeneous equations. Balasko *et al.* (1990) develop an ingenious technique to handle the unboundedness problem with link portfolios under linear portfolio constraints, but the approach is not applicable in the presence of nonlinear portfolio constraints. Angeloni and Cornet (2006), and Cornet and Gopalan (2010) consider various forms of convex portfolio constraints in multi-period stochastic exchange economies. Aouani and Cornet (2009) analyze polyhedral portfolio constraints.

2. MODEL

An economy with financial asset markets persists in two periods, 0 and 1. Financial markets are open in the first period (period 0) while markets for consumption goods are open in the second period (period 1). There are J financial assets, indexed by $j \in \mathcal{J} = \{1, 2, \dots, J\}$. Assets pay in monetary units in period 1. The payoffs are contingent on the realizations of uncertainty in period 1. The uncertainty to be resolved in period 1 is represented by S states of nature, indexed by $s \in \mathcal{S} = \{1, \dots, S\}$. Each asset $j \in \mathcal{J}$ pays $r_j(s)$ in state s . The payoffs of J assets in state s are given by the J -dimensional row vector $r(s) = (r_j(s))_{j \in \mathcal{J}}$, whereas the payoffs of asset j are summarized as the S -dimensional column vector $r_j = (r_j(s))_{s \in \mathcal{S}}$. The asset (payoff) structure is described by the $S \times J$ matrix $R = [(r(s))_{s \in \mathcal{S}}]$. Redundant assets exist when J is greater than the rank of R . There are L consumption goods in each state $s \in \mathcal{S}$. Since consumption is avail-

able only in period 1, the total number of commodities equals $\ell := LS$, implying that \mathbb{R}^ℓ becomes the commodity space of the economy.

The economy is populated by I agents, indexed by $i \in \mathcal{I} = \{1, 2, \dots, I\}$. Each agent $i \in \mathcal{I}$ has the consumption set X_i with the initial endowment $e_i \in X_i$ of commodities, and has a preference relation \succ_i on X_i , which induces a correspondence P_i on X_i such that for each $x_i \in X_i$, $P_i(x_i) = \{x'_i \in X_i : x'_i \succ_i x_i\}$. Portfolio choices of agents may be constrained by market frictions or legal arrangements. The portfolio constraint for agent i is represented by a set Θ_i in \mathbb{R}^J with $0 \in \Theta_i$. Agents are assumed to have zero initial endowments of marketed financial assets.

For a pair (p, q) in $\mathbb{R}^\ell \times \mathbb{R}^J$ and a point y in \mathbb{R}^ℓ , we follow the notational convention:

$$p \square_1 y := \begin{bmatrix} p(1) \cdot y(1) \\ \vdots \\ p(S) \cdot y(S) \end{bmatrix}, \quad p \square y := \begin{bmatrix} 0 \\ p \square_1 y \end{bmatrix}, \quad W(q) := \begin{bmatrix} -q \\ R \end{bmatrix}.$$

The vector $p \square_1 y$ indicates a stack of contingent expenditures on the consumption y of period 1. The first entry in $p \square y$ is zero because no consumption arises in period 0. Agent i 's open budget correspondence $\mathcal{B}_i : \mathbb{R}^\ell \times \mathbb{R}^J \rightarrow 2^{X_i \times \Theta_i}$ is defined by

$$\mathcal{B}_i(p, q) = \{(x_i, \theta_i) \in X_i \times \Theta_i : p \square (x_i - e_i) \ll W(q) \cdot \theta_i\},^4$$

while budget correspondence $c\ell\mathcal{B}_i : \mathbb{R}^\ell \times \mathbb{R}^J \rightarrow 2^{X_i \times \Theta_i}$ is defined by $c\ell\mathcal{B}_i(p, q) := c\ell[\mathcal{B}_i(p, q)]$.⁵ Agent i 's choice (x_i, θ_i) is \succ_i -maximal in $c\ell\mathcal{B}_i(p, q)$ if $(P_i(x_i) \times \Theta_i) \cap c\ell\mathcal{B}_i(p, q) = \emptyset$. We let $\mathcal{E} := \langle (X_i, \succ_i, e_i, \Theta_i)_{i \in \mathcal{I}}, R \rangle$ denote the economy described above.

Definition 2.1: A *competitive equilibrium* of economy \mathcal{E} is a profile $(p^*, q^*, x^*, \theta^*) \in \mathbb{R}^\ell \times \mathbb{R}^J \times \prod_{i \in \mathcal{I}} X_i \times \prod_{i \in \mathcal{I}} \Theta_i$ such that

⁴For two vectors v and v' in an Euclidean space, $v \geq v'$ implies that $v - v' \in \mathbb{R}_+^\ell$; $v > v'$ implies that $v \geq v'$ and $v \neq v'$; $v \gg v'$ implies that $v - v' \in \mathbb{R}_{++}^\ell$.

⁵For a nonempty subset A of an Euclidean space, we denote the closure of A by $c\ell A$ and the interior of A by $int A$.

- (i) $(x_i^*, \theta_i^*) \in cl\mathcal{B}_i(p^*, q^*), \forall i \in \mathcal{J}$,
- (ii) $(P_i(x_i^*) \times \Theta_i) \cap cl\mathcal{B}_i(p^*, q^*) = \emptyset, \forall i \in \mathcal{J}$,
- (iii) $\sum_{i \in \mathcal{J}} (x_i - e_i) = 0$ and $\sum_{i \in \mathcal{J}} \theta_i = 0$.

The results of the paper are built on the following five standard assumptions and two additional conditions to be discussed in Section 3.

- (A1)** For each $i \in \mathcal{J}$, $X_i = \mathbb{R}_+^\ell$.
- (A2)** Each P_i is strictly monotone, continuous, and convex-valued on \mathbb{R}_+^ℓ , and satisfies $x_i \notin P_i(x_i)$ for each $x_i \in \mathbb{R}_+^\ell$.⁶
- (A3)** Each e_i is in the interior of \mathbb{R}_+^ℓ .
- (A4)** Each Θ_i is a closed convex set in \mathbb{R}^J with $0 \in \Theta_i$.

Assumptions (A1)–(A3) are well-known conditions. Assumption (A4) can be used to model institutional or informational frictions that prevail in real-world financial markets. As illustrated in Luttmer (1996), it covers market frictions such as short-selling constraints, margin requirements, bid-ask spreads, and proportional transaction costs. Citanna and Villanacci (2000a,b) demonstrate that the measurability restrictions on optimal portfolios, which naturally occur in the asymmetric information model, are represented by linear portfolio restrictions.

3. LINK PORTFOLIOS AND CCIR CONDITION

3.1. LINK PORTFOLIOS AND EQUILIBRIUM ASSET PRICES

The source of link portfolios is redundant assets. Redundant assets exist when the number of assets is greater than the rank of R . Let $\langle R \rangle$ denote the

⁶Let x_i be a point in \mathbb{R}_+^ℓ . The preference ordering P_i is strictly monotone if $x_i + v \in P_i(x_i)$ for every $v \in \mathbb{R}_+^\ell \setminus \{0\}$, is continuous if both $P_i(x_i)$ and $P_i^{-1}(x_i) := \{x'_i \in X_i : x_i \succ_i x'_i\}$ are open, and is convex-valued if $P_i(x_i)$ is convex.

subspace spanned by the row vectors of the payoff matrix R and $\langle R \rangle^\perp$ be its kernel, i.e., $\langle R \rangle^\perp = \{\theta \in \mathbb{R}^J : R \cdot \theta = 0\}$. Portfolios in $\langle R \rangle^\perp$ are called *null-income portfolios*. By definition, they pay nothing in each state of the second period. The set $\langle R \rangle^\perp$ is a nontrivial subspace when redundant assets exist.

Definition 3.1: Let $\{\Theta'_i : i \in \mathcal{J}\}$ be a collection of closed convex sets in \mathbb{R}^J that satisfies $0 \in \Theta'_i$ for each $i \in \mathcal{J}$. A null-income portfolio η is called a *link portfolio* for $\{\Theta'_i : i \in \mathcal{J}\}$ if, for all $\lambda \in \mathbb{R}$, it satisfies $\lambda \eta \in \sum_{i \in \mathcal{J}} (\Theta'_i \cap \langle R \rangle^\perp)$.

Let $L(\Theta)$ denote the set of link portfolios for the portfolio constraints $\{\Theta_i : i \in \mathcal{J}\}$ and $M(\Theta)$ denote its orthogonal complement in $\langle R \rangle^\perp$. Then each $\theta \in \mathbb{R}^J$ has the orthogonal decomposition $\theta = \eta + (\theta - \eta)$ with $\eta \in L(\Theta)$ and $\theta - \eta \in \langle R \rangle + M(\Theta)$. The portfolio η is the *link component (or portfolio)* and $\theta - \eta$ is called the *income-determining component (or portfolio)* of θ . A portfolio in Θ_i gives the same level of contingent incomes in the future as its income-determining portfolio. The latter need not be feasible, however, under the portfolio constraint.

A link portfolio for $\{\Theta_i : i \in \mathcal{J}\}$ is an aggregate of individually feasible null-income portfolios that span a line in the set $\sum_{i \in \mathcal{J}} (\Theta_i \cap \langle R \rangle^\perp)$. Thus, the set of link portfolios $L(\Theta)$ is the maximal subspace in $\sum_{i \in \mathcal{J}} (\Theta_i \cap \langle R \rangle^\perp)$. For a convex set A in \mathbb{R}^J , let $\mathcal{K}(A)$ denote the recession cone of A and $\mathcal{L}(A)$ denote the maximal subspace in $A - \{a\}$ for some $a \in A$.⁷ It holds that $L(\Theta) = \mathcal{L}(\sum_{i \in \mathcal{J}} (\Theta_i \cap \langle R \rangle^\perp))$. For each $i \in \mathcal{J}$, let C_i denote the set $\mathcal{K}(\Theta_i)$.

Let $L(C)$ denote the set of link portfolios for $\{C_i : i \in \mathcal{J}\}$ and $M(C)$ denote the orthogonal complement of $L(C)$ in $\langle R \rangle^\perp$. By definition, it holds that $L(\Theta) + M(\Theta) = L(C) + M(C) = \langle R \rangle^\perp$. The portfolio space \mathbb{R}^J has the following decompositions

$$\mathbb{R}^J = \langle R \rangle + \langle R \rangle^\perp = \langle R \rangle + M(\Theta) + L(\Theta).$$

⁷The recession cone of the convex set A is defined as $\{v \in \mathbb{R}^m : A + v \subset A\}$. When A is closed, so is $\mathcal{K}(A)$. In this case, the recession cone $\mathcal{K}(A)$ is equivalent to the asymptotic cone of A and so it is expressed as $\mathcal{K}(A) = \{v \in \mathbb{R}^m : \exists \text{ sequences } \{x^n\} \text{ in } A \text{ and } \{a^n\} \text{ in } \mathbb{R}_{++} \text{ such that } a^n \rightarrow 0 \text{ and } a^n x^n \rightarrow v\}$. However, $\mathcal{K}(A)$ need not be closed if A is not closed and thus $\mathcal{K}(A) \neq \mathcal{K}(cl(A))$ in general. See Rockafellar (1970) for the properties of recession cone.

Since each C_i is in Θ_i , $L(C)$ is in $L(\Theta)$. It is worth noting that $L(\Theta) = \{0\}$ if and only if $L(C) = \{0\}$, and thus $L(C)$ is the source of link portfolios.

For each $i \in \mathcal{J}$, let $\bar{\Theta}_i$ denote the orthogonal projection of Θ_i onto $\langle R \rangle + M(\Theta)$. The set $\bar{\Theta}_i$ is the collection of income-determining components of portfolios in Θ_i . Note that $\bar{\Theta}_i$ need not be closed. Let \bar{C}_i denote the recession cone of $cl(\bar{\Theta}_i)$. To discuss the existence of equilibrium, we need to introduce the sets of viable prices. By Theorem 3.1, they must lie in $\langle R \rangle + M(\Theta)$. We define sets Q_i and Q of no-arbitrage asset prices by

$$Q_i = \{q \in \langle R \rangle + M(\Theta) : q \cdot \theta_i > 0 \text{ for all } \theta_i \in \bar{C}_i \text{ with } R \cdot \theta_i > 0\}$$

and $Q = \bigcap_{i \in \mathcal{J}} Q_i$.⁸

We recall that no consumption arises in period 0. Thus, equilibrium may fail to exist if portfolio constraints are so tight that agents cannot trade assets to make positive income transfer to future states. To avoid the problem, we need to impose the following condition.

(A5) For each $i \in \mathcal{J}$, there exists $v_i \in C_i$ such that $R \cdot v_i > 0$.

(A6) For each $q \in cl(Q) \setminus \{0\}$ and for each $i \in \mathcal{J}$, there exists $\theta_i \in \Theta_i$ such that $q \cdot \theta_i < 0$.

Assumption (A5) ensures that no satiation occurs in the case without consumption in period 0. It is illustrated in Polemarchakis and Siconolfi (1993) that no equilibrium exists in a one-good economy which fails to satisfy Assumption (A5) because agents have a satiation portfolio in the reduced-form economy.⁹ Assumption (A6) combined with Assumption (A3) constitutes a survival condition for constrained financial markets. They ensure the presence of consumptions which are cheaper than attainable incomes in each contingency in equilibrium.

⁸As shown in Won and Hahn (2007), prices in Q_i are ‘arbitrage-free’ with respect to so-called ‘projective arbitrage’. The no projective arbitrage condition is necessary for the existence of equilibrium in constrained financial markets with derivatives.

⁹Assumption (A5) can be dispensed with in the case that preferences are strictly monotone and consumption is available in the initial period.

The following shows that link portfolios have null value in equilibrium.

Theorem 3.1: Suppose Assumptions (A1), (A2), (A4), and (A5) hold. If q is an equilibrium asset price of \mathcal{E} , then $q \cdot \eta = 0$ for all $\eta \in L(\Theta)$.

PROOF : See Appendix. ■

Theorem 3.1 implies that equilibrium asset prices are orthogonal to $L(\Theta)$, i.e., they are in $\langle R \rangle + M(\Theta)$. The property of equilibrium asset prices plays an important role in verifying the existence of equilibrium.

3.2. ECONOMIES WITHOUT NONZERO LINK PORTFOLIOS

As a preliminary step, we provide the existence of equilibrium in the case that the economy \mathcal{E} is free from nonzero link portfolios. That is, in this section we suppose temporarily that $L(\Theta) = \{0\}$. The preliminary result will be used to verify the existence of equilibrium for the economy with nonzero link portfolios in Section 5.

Theorem 3.2: Suppose that $L(\Theta) = \{0\}$. Then economy \mathcal{E} has an equilibrium under Assumptions (A1)–(A6).

PROOF : See Appendix. ■

When $L(\Theta) \neq \{0\}$, equilibrium may fail to exist as illustrated later in Section 5. The following section present conditions which ensure the existence of equilibrium in the presence of nonzero link portfolios.

3.3. ECONOMIES WITH LINK PORTFOLIOS

For a contingent-income vector $\omega_i \in \mathbb{R}^S$ and a nonempty set A in \mathbb{R}^J , we define a set $A(\omega_i) := \{\theta \in A : R \cdot \theta = \omega_i\} = A \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}$. Then the set $\Theta_i(\omega_i)$ consists of portfolios in Θ_i which yield the contingent incomes ω_i in period 1. The following provides a formal definition of contingent income redistribution.

Definition 3.2: A profile of contingent income vectors $\omega = (\omega_1, \omega_2, \dots, \omega_I)$ in $(\mathbb{R}^S)^I$ is called a *contingent income redistribution (CIR)* if $\sum_{i \in \mathcal{J}} \omega_i = 0$ and $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i) \neq \emptyset$.

For a CIR $\omega = (\omega_1, \omega_2, \dots, \omega_I)$, let θ be a point in $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$. Then there exists $\theta_i \in \Theta_i$ for each $i \in \mathcal{J}$ such that $R \cdot \theta_i = \omega_i$ and $\theta = \sum_{i \in \mathcal{J}} \theta_i$. Since $\sum_{i \in \mathcal{J}} R \cdot \theta_i = \sum_{i \in \mathcal{J}} \omega_i = 0$, the set $\{R \cdot \theta_i : i \in \mathcal{J}\}$ forms a redistribution of contingent incomes over agents that is attained by individually feasible portfolios. Thus, aggregate feasible portfolios in $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$ induce the CIR ω . Notice that the CIR-inducing portfolio allocation $(\theta_1, \dots, \theta_I)$ need not be market-clearing.¹⁰ We impose the condition that the set of CIR-inducing aggregate feasible portfolios be closed in the portfolio space.

(A7) For a CIR $\omega = (\omega_1, \omega_2, \dots, \omega_I)$ in $(\mathbb{R}^S)^I$, the following hold.

- (i) (CCIR) $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$ is closed in \mathbb{R}^J , i.e., $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i) = c\ell(\sum_{i \in \mathcal{J}} \Theta_i(\omega_i))$.
- (ii) $L(\Theta) = \mathcal{L}(\sum_{i \in \mathcal{J}} \Theta_i(\omega_i))$.
- (iii) For each $i \in \mathcal{J}$, $\sum_{i \in \mathcal{J}} (c\ell(\overline{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}) \subset \sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$.

The first condition of Assumption (A7) is the CCIR condition discussed in Introduction. The condition trivially holds when there is no nonzero link portfolio, i.e., $L(\Theta) = \{0\}$.¹¹ The CCIR condition states that the limit of a convergent sequence of aggregate feasible portfolios inducing the CIR ω is also an aggregate feasible portfolio. The condition is a key condition of the paper and, to the best of our knowledge, is new to the literature. Especially, the CCIR condition is needed to build the fundamental theorem of portfolio decomposition by which optimal portfolios are split into link components and income-determining components. As demonstrated in Section 6, Assumption (A7) encompasses existing conditions on portfolio constraints which are studied in the recent literature such as Aouani and Cornet (2009, 2011) and Martins-da-Rocha and Triki (2005).

¹⁰Since R does not have full rank and a CIR ω requires $\sum_i R \cdot \theta_i = \sum_i \omega_i = 0$, we have $\sum_i \theta_i \in \langle R \rangle^\perp$.

¹¹This is a special case of (C2) discussed in Section 6.

The conditions (ii) and (iii) of Assumption (A7) are a more or less technical one which complements the CCIR condition to validate the fundamental theorem of portfolio decomposition. The condition (ii) requires that the set of link portfolios coincide with the maximal subspace of aggregate feasible portfolios inducing each CIR. The condition (iii) is introduced to deal with the problematic case where $\bar{\Theta}_i$'s are not closed in \mathbb{R}^J . This condition is unnecessary if each $\bar{\Theta}_i$ is closed in \mathbb{R}^J .

4. THE FUNDAMENTAL THEOREM OF PORTFOLIO DECOMPOSITION

This section provides the fundamental theorem of portfolio decomposition on which the main result of the paper are built in the next section. For each $\omega_i \in \mathbb{R}^S$, we recall that $\bar{\Theta}_i(\omega_i) = \{\theta \in \bar{\Theta}_i : R \cdot \theta = \omega_i\}$, which is the orthogonal projection of $\Theta_i(\omega_i)$ onto $\langle R \rangle + M(\Theta)$. This set does not contain nonzero link portfolios and generates the same set of contingent incomes as $\Theta_i(\omega_i)$. A difficulty with $\bar{\Theta}_i(\omega_i)$ is that portfolios in $\bar{\Theta}_i$ need not be feasible, i.e., $\bar{\Theta}_i \not\subset \Theta_i$ and moreover, $\bar{\Theta}_i$ need not be closed. The fundamental theorem of portfolio decomposition clarifies the relationship between $\Theta_i(\omega_i)$'s and $\bar{\Theta}_i(\omega_i)$'s which plays a key role in investigating the existence of equilibrium in the economy \mathcal{E} .

The following result provides a way in which a portfolio allocation is orthogonally decomposed into the link portfolio and the allocation of income-determining portfolios.

Proposition 4.1 (Fundamental Theorem of Portfolio Decomposition): Let $\omega = (\omega_1, \omega_2, \dots, \omega_J)$ be a CIR in $(\mathbb{R}^S)^J$. Under Assumptions (A4) and (A7), the following hold.

- (1) $\sum_{i \in J} \Theta_i(\omega_i) = L(\Theta) + \sum_{i \in J} \bar{\Theta}_i(\omega_i)$.
- (2) $\sum_{i \in J} \bar{\Theta}_i(\omega_i)$ is closed in \mathbb{R}^J .
- (3) $\sum_{i \in J} \bar{\Theta}_i(\omega_i) = \sum_{i \in J} c\ell(\bar{\Theta}_i(\omega_i)) = \sum_{i \in J} (c\ell(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\})$.
- (4) $\mathcal{L}(\sum_{i \in J} (\bar{C}_i \cap \langle R \rangle^\perp)) = \{0\}$.

PROOF : See Appendix. ■

Recall that each aggregate feasible portfolio in $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$ induces the re-distribution $\omega = (\omega_1, \dots, \omega_I)$ of contingent incomes among agents. The first result (1) of Proposition 4.1 states that each aggregate feasible portfolio inducing the CIR ω under the portfolio constraints $\{\Theta_i\}$ is decomposed into the link portfolio and the aggregate income-determining portfolio. The second result (2) shows that Assumption (A7) ensures the closedness of $\sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i)$ where each $\bar{\Theta}_i(\omega_i)$ need not be closed. The third result (3) shows that $\bar{\Theta}_i(\omega_i)$ and $cl(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}$ are interchangeable at the aggregate level. The last result is equivalent to the relation

$$\mathcal{L} \left(\sum_{i \in \mathcal{J}} (cl(\bar{\Theta}_i) \cap \langle R \rangle^\perp) \right) = \{0\}.$$

Thus, it states that the set of portfolio constraints $\{cl(\bar{\Theta}_i), i \in \mathcal{J}\}$ has no nonzero link portfolio. In the next section, the above result will provide a foothold for verifying the existence of equilibrium for the economy \mathcal{E} by exploiting the result of Theorem 3.2.

5. MAIN RESULTS

This section verifies the existence of equilibrium for the economy \mathcal{E} . To do this, we introduce the *projected economy* $\bar{\mathcal{E}}$ which is identical to the original economy \mathcal{E} except that Θ_i is replaced by $cl(\bar{\Theta}_i)$ for each $i \in \mathcal{J}$.¹² A competitive equilibrium of $\bar{\mathcal{E}}$ can be defined in the same fashion as that of \mathcal{E} . The projected economy $\bar{\mathcal{E}}$ shares the same equilibrium prices and consumption allocations with the original economy \mathcal{E} and is much easier to deal with than \mathcal{E} because nonzero link portfolios do not appear in the portfolio constraints of $\bar{\mathcal{E}}$. To discuss the property of equilibrium prices, we introduce the set of normalized prices $\Delta = \Delta_1 \times \Delta_0$ where

$$\Delta_0 = \{q \in \mathcal{Q} : \|q\| = 1\}, \quad \Delta_s = \{p \in \mathbb{R}_{++}^L : \sum_{k=1}^L p_k(s) = 1\}, \quad \Delta_1 = \prod_{s \in \mathcal{S}} \Delta_s.$$

¹²The set $cl(\bar{\Theta}_i)$ is taken as the portfolio constraint for agent i in $\bar{\mathcal{E}}$ instead of $\bar{\Theta}_i$ because $\bar{\Theta}_i$ need not be closed.

The following result shows that both \mathcal{E} and $\overline{\mathcal{E}}$ have the same equilibrium outcomes except for the possible presence of nonzero link components in optimal portfolios for \mathcal{E} . This result is exploited to verify the existence of equilibrium for the economy \mathcal{E} .

Proposition 5.1: The following hold true under Assumptions (A1)–(A7).

- (i) If $\overline{\mathcal{E}}$ has an equilibrium $(p^*, q^*, x^*, \overline{\theta}^*) \in \Delta \times \prod_{i \in \mathcal{J}} X_i \times \prod_{i \in \mathcal{J}} cl(\overline{\Theta}_i)$, then there exists $\theta^* \in \Theta_i$ for each $i \in \mathcal{J}$ such that $(p^*, q^*, x^*, \theta^*)$ is an equilibrium of \mathcal{E} .
- (ii) If \mathcal{E} has an equilibrium $(p^*, q^*, x^*, \theta^*) \in \Delta \times \prod_{i \in \mathcal{J}} X_i \times \prod_{i \in \mathcal{J}} \Theta_i$, then $\overline{\mathcal{E}}$ has an equilibrium $(p^*, q^*, x^*, \overline{\theta}^*)$ where $\overline{\theta}_i^*$ is the projection of θ_i^* onto $\langle R \rangle + M(\Theta)$.

PROOF : See Appendix. ■

We already know from Theorem 3.1 that link portfolios are free in equilibrium of the economy \mathcal{E} . Proposition 5.1 goes further by showing that both \mathcal{E} and $\overline{\mathcal{E}}$ share the same equilibrium profiles except for the link portfolios in individual optimal portfolios. This result is not intuitively obvious because each optimal portfolio $\overline{\theta}_i^*$ in the projected economy $\overline{\mathcal{E}}$ need not be feasible in the economy \mathcal{E} . The fundamental theorem of portfolio decomposition (Proposition 4.1) provides an answer to the linkage between equilibrium portfolio allocations in \mathcal{E} and $\overline{\mathcal{E}}$. By (1) and (3) of Proposition 4.1, there exists a link portfolio which transforms the set of individual optimal portfolio $\overline{\theta}_i^*$'s in the projected economy $\overline{\mathcal{E}}$ into the feasible portfolios θ_i^* 's in the economy \mathcal{E} that yield the same contingent incomes as $\overline{\theta}_i^*$'s. Since the link component of θ_i^* is free for each $i \in \mathcal{J}$, the equilibrium prices and optimal consumptions of $\overline{\mathcal{E}}$ are kept in equilibrium of the economy \mathcal{E} . The same intuition can be exerted to show the validity of the converse.

Theorem 3.2 combined with Proposition 5.1 leads to the existence of equilibrium in the economy \mathcal{E} with possibly nonzero link portfolios.

Theorem 5.1: Under Assumptions (A1)–(A7), there exists an equilibrium in the economy \mathcal{E} .

PROOF : First, we want to apply Theorem 3.2 to the projected economy $\bar{\mathcal{E}}$. To do this, we need to check that Assumptions (A1)–(A6) hold in $\bar{\mathcal{E}}$. Assumptions (A1)–(A4) trivially holds in $\bar{\mathcal{E}}$. By Assumption (A5), we choose $v_i \in C_i$ with $R \cdot v_i > 0$. Let \bar{v}_i denote the projection of v_i onto $\langle R \rangle + M(\Theta)$. Then $\bar{v}_i \in cl(\bar{\Theta}_i)$ and $R \cdot \bar{v}_i = R \cdot v_i > 0$. Thus, Assumption (A5) is fulfilled in $\bar{\mathcal{E}}$. Now by Assumption (A6), for each $q \in cl(Q) \setminus \{0\}$, we choose $\zeta_i \in \Theta_i$ such that $q \cdot \zeta_i < 0$. Let $\bar{\zeta}_i$ denote the projection of ζ_i onto $\langle R \rangle + M(\Theta)$. By Theorem 3.1, we have $q \cdot \bar{\zeta}_i = q \cdot \zeta_i < 0$. That is, Assumption (A6) holds in $\bar{\mathcal{E}}$. Consequently, the economy $\bar{\mathcal{E}}$ satisfies Assumptions (A1)–(A6). On the other hand, by (4) of Proposition 4.1, $\bar{\mathcal{E}}$ has no nonzero link portfolios. Then it follows from Theorem 3.2 that the economy $\bar{\mathcal{E}}$ has an equilibrium. This result combined with Proposition 5.1 ensures the existence of equilibrium in the economy \mathcal{E} . ■

In fact, Theorem 5.1 is built on the following procedure.

Four-Step Procedure for the Existence of Equilibrium:

- (S1) Verify the existence of equilibrium for a special economy where the portfolio constraints admit no nonzero link portfolio.
- (S2) Find conditions on the original portfolio constraints under which the fundamental theorem of portfolio decomposition holds.
- (S3) Construct the artificial economy $\bar{\mathcal{E}}$ where agent i faces the portfolio constraint $cl(\bar{\Theta}_i)$ and check that the artificial economy is qualified as the special economy mentioned in Step (S1).
- (S4) Verify the existence of equilibrium for the original economy by checking the allocational equivalence between the original economy and the artificial economy.

Let us check that our approach for verifying the result of Theorem 5.1 is in conformity with the four-step procedure. Step (S1) is done in Theorem 3.2

and Assumption (A7) is our choice in Step (S2). By (4) of Proposition 4.1, the portfolio constraints admit no nonzero link portfolios in the artificial economy $\bar{\mathcal{E}}$ and thus, Step (S3) is fulfilled. Finally, Proposition 5.1 provides an answer to Step (S4).

Step (S2) is the most critical one of the four-step procedure. Distinct approaches of the literature can be articulated in terms of the above procedure. Specifically, the four-step procedure can be specialized in Martins-da-Rocha and Triki (2005) and Aouani and Cornet (2011), which make distinct assumptions on portfolio restrictions in Step (S2). As shown in the next section, the current paper makes the most comprehensive assumption (Assumption (A7)) in the literature for Step (S2).

Finally, we provide an example in which the economy satisfies all the standard conditions but fails to have equilibrium only because the CCIR condition is violated. The failure of the CCIR condition is attributed to the presence of a nonzero link portfolio in the example.

Example 5.1: We consider a single good exchange economy with 2 agents, 4 assets, and 3 states. The single good is used as a numéraire. We assume that the payoff matrix R is given as

$$R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & 1 & 3 \end{bmatrix}.$$

Both agents have the same endowment of goods and distinct preferences.

$$\begin{aligned} u_1(x) &= x(1) + 0.5x(2) + 0.5x(3), & e_1 &= (1, 1, 1), \\ u_2(x) &= x(1) + x(2) + x(3), & e_2 &= (1, 1, 1). \end{aligned}$$

We assume that

$$\begin{aligned} \Theta_1 &= \{(a, b, c, d) \in \mathbb{R}^4 : b \geq -1/2, c \geq -2b + 1/(b+1) - 4d - 2\}, \\ \Theta_2 &= \{(a, b, c, d) \in \mathbb{R}^4 : b \leq 1, c \geq -2b - 3d - 1, d \geq -2\}. \end{aligned}$$

Portfolio $\eta = (2, 1, -2, 0)$ is a link portfolio of the economy since $\theta_1 + \lambda \eta \in \Theta_1$ for each $\theta_1 \in \Theta_1$ and $\lambda \geq 0$, and $\theta_2 + \lambda \eta \in \Theta_2$ for each $\theta_2 \in \Theta_2$ and $\lambda < 0$.

As shown in the appendix, the economy has no equilibrium because the link portfolio prevents the set $\sum_{i=1}^2 \Theta_i(0) = (\Theta_1 \cap \langle R \rangle^\perp) + (\Theta_2 \cap \langle R \rangle^\perp)$ from being closed in \mathbb{R}^4 , i.e., it causes the failure of the CCIR condition. Remarkably, each Θ_i and the marketed income set $Y_i := \{R \cdot \theta_i : \theta_i \in \Theta_i\}$ are closed and moreover, the aggregate income set $Y_1 + Y_2$ is closed, where $Y_1 = \{(y_1, y_2, y_2) \in \mathbb{R}^3 : y_2 = y_3\}$ and $Y_2 = \{(y_1, y_2, y_2) \in \mathbb{R}^3 : y_2 = y_3 \text{ and } y_3 \geq -1\}$. Such requirements do not guarantee the existence of equilibrium in this example. An exhaustive analysis of the example is relegated to Appendix. \square

Remark 5.1: The basic requirement that either Θ_i or $Y_i := \{R \cdot \theta_i : \theta_i \in \Theta_i\}$ be closed for each $i \in \mathcal{J}$ is taken as a standard condition in the literature.¹³ As shown in Example 5.1, however, such standard requirements are not sufficient for the existence of equilibrium, especially when nonzero link portfolios exist. Moreover, Example 5.1 illustrates that the additional requirement that the aggregate marketed income set $\sum_{i \in \mathcal{J}} Y_i$ be closed in \mathbb{R}^S may not be helpful in ensuring the existence of equilibrium in constrained financial markets. Assumption (A7) is introduced to capture the elusive behavior of link portfolios in equilibrium of constrained financial markets through portfolio decomposition. \square

6. A COMPARATIVE REVIEW OF PORTFOLIO RESTRICTIONS

The following provides a collection of the restrictions the literature imposes on portfolio constraints. Assumption (A7) turns out to be the most comprehensive assumption that encompasses all the existing conditions of the literature.

(C1) For each $i \in \mathcal{J}$, $C_i \cap \langle R \rangle^\perp = \{0\}$.

(C2) The collection $\{C_i \cap \langle R \rangle^\perp : i \in \mathcal{J}\}$ is positively semi-independent.¹⁴

(C3) Each Θ_i is a subspace of \mathbb{R}^J .

(C4) If $v_i \in C_i \cap \langle R \rangle^\perp$ for each $i \in \mathcal{J}$ and $\sum_{i \in \mathcal{J}} v_i = 0$, then $v_i \in -C_i$ for all $i \in \mathcal{J}$.

¹³The closedness of Y_i 's in \mathbb{R}^S is assumed in the literature of asset pricing such as Luttmer (1996) and Jouini and Kallal (1999).

¹⁴A set of cones $\{A_i : i \in \mathcal{J}\}$ is positively semi-independent if $v_i \in A_i$ for all $i \in \mathcal{J}$ and $\sum_{i \in \mathcal{J}} v_i = 0$ imply that $v_i = 0$ for all $i \in \mathcal{J}$.

(C5) Each Θ_i is a polyhedral convex set in \mathbb{R}^J , i.e., for each i , there exist a $k_i \times J$ matrix B_i and a vector $a_i \in \mathbb{R}^{k_i}$ for some positive integer k_i such that $\Theta_i = \{\theta \in \mathbb{R}^J : B_i \cdot \theta \leq a_i\}$.

(C6) The set $G = \{(\omega, v) \in (\mathbb{R}^S)^J \times \mathbb{R}^J : v \in \sum_{i \in \mathcal{J}} \Theta_i(\omega_i)\}$ is closed.

(C7) Each Θ_i satisfies the following relations:

(C7-1) For any $\theta \in \mathbb{R}^J$, there exist $i \in \mathcal{J}$ and $\alpha > 0$ such that $R \cdot \theta = \alpha R \cdot \theta$ for some $\theta_i \in \Theta_i$.¹⁵

(C7-2) $(-\sum_{i \in \mathcal{J}} \Theta_i) \cap \langle R \rangle^\perp \subset \sum_{i \in \mathcal{J}} (C_i \cap \langle R \rangle^\perp)$.

Condition (C6) is introduced in Aouani and Cornet (2011) while (C7) in Martins-da-Rocha and Triki (2005) and Cornet and Gopalan (2010). Aouani and Cornet (2011) show that (C1)–(C5) imply (C6). As illustrated below, however, (C6) and (C7) do not imply each other. As depicted below in a diagram, (C7) has the distinct property which is not shared with other portfolio conditions such as (C4), (C5) and (C6). It is shown below that Assumption (A7) encompasses both (C6) and (C7).

Condition (C1) is assumed in Siconolfi (1989). This condition is a special case of (C2). Portfolio constraints are represented by a system of homogeneous linear equations under the condition (C3) which is assumed in Balasko *et al.* (1990). Condition (C4) subsumes (C2) and (C3) as a special case.¹⁶ Condition (C5) is considered in Aouani and Cornet (2009). Clearly, (C3) is a special case of (C5). Either (C4) or (C5) does not imply (C7),¹⁷ and (C7) does not imply either (C4) or (C5).¹⁸ Since (C6) and (C7) implies Assumption (A7), it encompasses all the conditions mentioned above.

¹⁵This condition is called locally collectively frictionless in Martins-da-Rocha and Triki (2005).

¹⁶Notice that $\Theta_i = C_i = \mathcal{L}(\Theta_i)$ when Θ_i is a subspace.

¹⁷For example, suppose that $I = J = 2$, $S = 1$, $R = [0 \ 1]$, $\Theta_1 = \{(a, b) \in \mathbb{R}^2 : a = b\}$, and $\Theta_2 = \{(a, b) \in \mathbb{R}^2 : a = -b\}$. Clearly, $\langle R \rangle^\perp = \{(a, b) \in \mathbb{R}^2 : b = 0\}$. Then, (C4) and (C5) hold trivially. But, (C7) does not hold because $-(\Theta_1 + \Theta_2) \cap \langle R \rangle^\perp = \langle R \rangle^\perp$ and $(C_1 \cap \langle R \rangle^\perp) + (C_2 \cap \langle R \rangle^\perp) = \{0\}$.

¹⁸Suppose that $I = J = 2$, $\Theta_1 = \mathbb{R}_+^2 \cup \{(a, b) \in \mathbb{R}^2 : b \geq a^2\}$, $\Theta_2 = \mathbb{R}^2$, and $\langle R \rangle^\perp = \{(a, b) \in \mathbb{R}^2 : b = 0\}$. Then we have $C_1 = \mathbb{R}_+^2$ and $C_2 = \mathbb{R}^2$. Clearly, (C7) holds and (C4) does not hold. Since Θ_1 is not polyhedral, (C5) does not hold either.

The following propositions show that both (C6) and (C7) imply Assumption (A7).

Proposition 6.1: Condition (C6) implies Assumption (A7).

PROOF : See Appendix. ■

Proposition 6.2: Condition (C7) implies Assumption (A7).

PROOF : See Appendix. ■

The following corollary is immediate from Theorem 5.1 and the above discussions and propositions.

Corollary 6.1: Suppose that Assumptions (A1)–(A6) hold and (A7) is replaced by one of the conditions (C1)–(C7). Then the economy \mathcal{E} has an equilibrium.

We provide an example where (C7) holds but (C6) fails. Thus, (C7) does not imply (C6). Moreover, by Proposition 6.2, Assumption (A7) is fulfilled in the example and therefore the converse of Proposition 6.1 does not hold.

Example 6.1: We consider a single good exchange economy with 2 agents, 2 assets, and 1 state. We assume that the payoff matrix $R = [1 \ 0]$ and agents have the portfolio choice set as follows:

$$\begin{aligned}\Theta_1 &= \{(a_1, b_1) \in \mathbb{R}^2 : b_1 \geq -1 + 1/(1 - a_1), a_1 < 1\}, \\ \Theta_2 &= \{(a_2, b_2) \in \mathbb{R}^2 : a_2 \geq -1\}.\end{aligned}$$

We see that $C_1 = \{(a_1, b_1) \in \mathbb{R}^2 : a_1 \leq 0, b_1 \geq 0\}$, $C_2 = \{(a_2, b_2) \in \mathbb{R}^2 : a_2 \geq 0\}$ and $L(\Theta) = \langle R \rangle^\perp = \{(a, b) \in \mathbb{R}^2 : a = 0\}$. Since $\langle R \rangle^\perp \subset -\Theta_2$ and $\langle R \rangle^\perp \subset C_2$, it holds that

$$-(\Theta_1 + \Theta_2) \cap \langle R \rangle^\perp = \langle R \rangle^\perp = (C_1 \cap \langle R \rangle^\perp) + (C_2 \cap \langle R \rangle^\perp).$$

Thus, (C7-2) holds in the example. To check the validity of (C7-1), we pick any $\theta = (a, b) \in \mathbb{R}^2$ with $R \cdot \theta = a$. If $a < 0$, we can pick $\theta_1 = (a, 0) \in \Theta_1$ which

gives $R \cdot \theta_1 = a$. If $a \geq 0$, we can pick $\theta_2 = (a, 0) \in \Theta_2$ which gives $R \cdot \theta_2 = a$. Thus, (C7-1) holds here as well. By Proposition 6.2, Assumption (A7) holds too.

However, (C6) fails here. To see this, we take a sequence $\{(\omega^n, v^n)\}$ where

$$\begin{aligned} \theta^n &= (\theta_1^n, \theta_2^n) = ((1 - 1/n, n), (0, -n)) \in \Theta_1(\omega_1^n) \times \Theta_2(\omega_2^n), \\ \omega^n &= (R \cdot \theta_1^n, R \cdot \theta_2^n) = (1 - 1/n, 0), \\ v^n &= \theta_1^n + \theta_2^n = (1 - 1/n, n) + (0, -n) = (1 - 1/n, 0) \in \sum_{i=1}^2 \Theta_i(\omega_i^n). \end{aligned}$$

Note that $\omega^n \rightarrow \omega^* = (1, 0)$ and $v^n \rightarrow v^* = (1, 0)$. As $\Theta_1(1) = \emptyset$, it holds that

$$v^* = (1, 0) \notin \sum_{i=1}^2 \Theta_i(\omega_i^*) = \Theta_1(1) + \Theta_2(0).$$

Hence (C6) does not hold in this economy. Consequently, (C7) does not imply (C6) and the converse of Proposition 6.1 is not true. \square

By using an arrow as the implicative direction, the relationship among (C1)–(C7) and Assumption (A7) is summarized as follows.

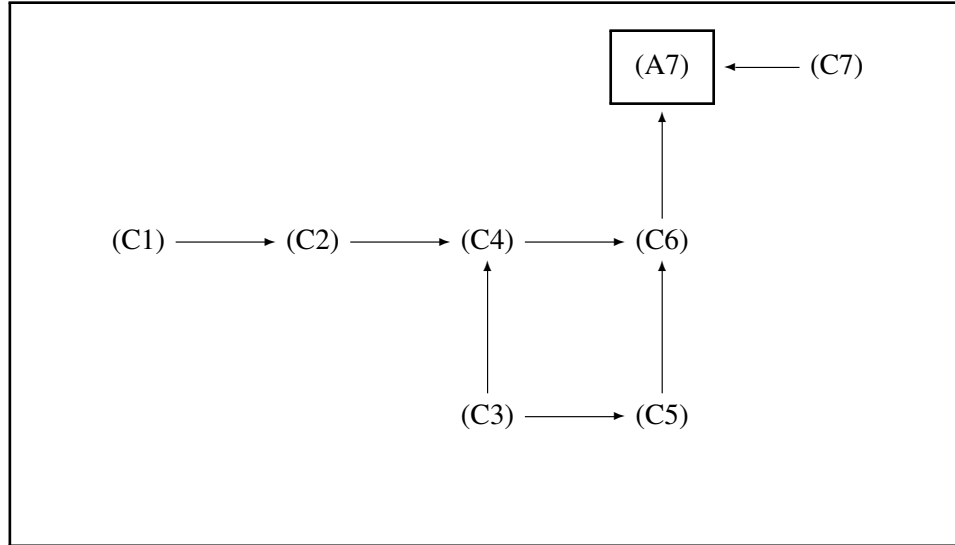


Figure 1. The Relationship between the Portfolio Restrictions.

7. CONCLUSION

The paper has developed the four-step modular approach to study the effect of portfolio constraints on the existence of equilibrium by building the two pillars to deal with link portfolios. The two pillars are the fundamental theorem of portfolio decomposition and the allocational equivalence between the original economy and the artificial economy where each agent are restrained to make portfolio choices in the closure of income-determining components of originally feasible portfolios. The fundamental theorem of portfolio decomposition is built on Assumption (A7) where the CCIR condition plays a pivotal role. Assumption (A7) encompasses all the distinct restrictions on constrained portfolios which appear in the literature.

Portfolio constraints arise from many sources of market frictions such as institutional imperfections, transaction costs, and informational asymmetry. The property of link portfolios may be useful in studying the role of financial derivatives from a perspective of alternative market frictions. Financial derivatives affect individual welfare in frictional markets because they expand risk-sharing opportunities. Thus, it will be an interesting research to analyze the welfare implications of link portfolios in frictional markets. The result of the paper is potentially applicable to the literature on information revelation. Since link portfolios represent linear restrictions at the aggregate level, the fundamental theorem of portfolio decomposition may contribute to the extension of Citanna and Villanacci (2000a,b) to the case of constrained financial markets.

APPENDIX

A.1. Proof of Theorem 3.1

Let (p, q, x, θ) be an equilibrium of \mathcal{E} . Suppose that there exists $\eta \in L(\Theta)$ with $q \cdot \eta \neq 0$. Since $-\eta \in L(\Theta)$, without loss of generality, we may assume that $q \cdot \eta < 0$. For each $\lambda > 0$ and $i \in \mathcal{I}$, we pick $\eta_i(\lambda) \in \Theta_i \cap \langle R \rangle^\perp$ such that $\lambda^2 \eta = \sum_{i \in \mathcal{I}} \eta_i(\lambda)$. Since $\lim_{\lambda \rightarrow \infty} \lambda q \cdot \eta = \lim_{\lambda \rightarrow \infty} \sum_{i \in \mathcal{I}} q \cdot (\eta_i(\lambda)/\lambda) = -\infty$, there exists

$i \in \mathcal{J}$ such that $\lim_{\lambda \rightarrow \infty} q \cdot (\eta_i(\lambda)/\lambda) = -\infty$. Without loss of generality, we will assume that $i = 1$.

By Assumption (A5), let v_1 be a point in C_1 with $R \cdot v_1 > 0$. Then there exists $\delta > 0$ in \mathbb{R}^ℓ such that $x_1 + \delta \succ_1 x_1$ and $p \square (x_1 + \delta - e_1) < R \cdot (\theta_1 + v_1)$. By the continuity of \succ_1 , there exists $\lambda_1 > 1$ such that for all $\lambda > \lambda_1$,

$$\left(1 - \frac{1}{\lambda}\right)(x_1 + \delta) \succ_1 x_1.$$

It is obvious that for all $\lambda > 1$, $p \square_1 \left[\left(1 - \frac{1}{\lambda}\right)(x_1 + \delta) - e_1\right] < R \cdot \left[\left(1 - \frac{1}{\lambda}\right)(\theta_1 + v_1)\right]$. Since $\lim_{\lambda \rightarrow \infty} q \cdot [\eta_1(\lambda)/\lambda] = -\infty$, for all $\lambda > \lambda_2$, there exists $\lambda_2 > 1$ such that $q \cdot \left[\left(1 - \frac{1}{\lambda}\right)(\theta_1 + v_1) + \frac{1}{\lambda}\eta_1(\lambda)\right] < 0$. Recalling that $\eta_1(\lambda) \in \langle R \rangle^\perp$, we have

$$R \cdot \left[\left(1 - \frac{1}{\lambda}\right)(\theta_1 + v_1) + \frac{1}{\lambda}\eta_1(\lambda)\right] = R \cdot \left[\left(1 - \frac{1}{\lambda}\right)(\theta_1 + v_1)\right].$$

For all $\lambda > \max\{\lambda_1, \lambda_2\}$, it follows that $\left(1 - \frac{1}{\lambda}\right)(x_1 + \delta) \succ_1 x_1$ and

$$\left(\left(1 - \frac{1}{\lambda}\right)(x_1 + \delta), \left(1 - \frac{1}{\lambda}\right)(\theta_1 + v_1) + \frac{1}{\lambda}\eta_1(\lambda)\right) \in c\ell\mathcal{B}_1(p, q).$$

This contradicts the optimality of (x_1, θ_1) in $c\ell\mathcal{B}_1(p, q)$. Thus, we conclude that $q \cdot \eta = 0$ for all $\eta \in L(\Theta)$. \blacksquare

A.2. Proof of Theorem 3.2

We introduce a price set $\tilde{\Delta} = \tilde{\Delta}_1 \times \tilde{\Delta}_0$ where

$$\tilde{\Delta}_0 = \{q \in \mathbb{R}^J : \|q\| \leq 1\}, \quad \tilde{\Delta}_s = c\ell(\Delta_s), \quad \tilde{\Delta}_1 = \prod_{s \in \mathcal{S}} \tilde{\Delta}_s.$$

Clearly, the set $\tilde{\Delta}$ is nonempty, compact, and convex.

Let $A := \{x \in \prod_{i \in \mathcal{J}} X_i : \sum_{i \in \mathcal{J}} (x_i - e_i) = 0\}$ and \hat{X}_i be the projection of A onto X_i . Since each X_i is bounded from below, \hat{X}_i is compact for all $i \in \mathcal{J}$. Thus we can consider a compact convex cube $K \subset \mathbb{R}^\ell$ with center 0 such that $\sum_{i \in \mathcal{J}} \hat{X}_i \subset \text{int}(K)$. Note that $\sum_{i \in \mathcal{J}} e_i \in \text{int}(K)$. Next we take an increasing sequence $\{M_n\}$ in \mathbb{R}^J of compact convex cubes with center 0 such that $0 \in \text{int}(M_1)$ and $\bigcup_n M_n = \mathbb{R}^J$. For each i , let $\tilde{X}_i := X_i \cap K$ and $\Theta_i^n := \Theta_i \cap M_n$, and define a new preference correspondence $\tilde{P}_i : \tilde{X}_i \rightarrow 2^{\tilde{X}_i}$ by $\tilde{P}_i(x_i) := P_i(x_i) \cap \tilde{X}_i$.

Consider the sequence of truncated economies $\{\mathcal{E}^n := \langle (\tilde{X}_i, \tilde{P}_i, e_i, \Theta_i^n)_{i \in \mathcal{J}}, R \rangle\}$. In the economy \mathcal{E}^n , each agent i has a nonempty compact convex choice set $\tilde{X}_i \times \Theta_i^n$. Let us define agent i 's modified open budget correspondence $\mathcal{B}_i^n : \tilde{\Delta} \rightarrow 2^{\tilde{X}_i \times \Theta_i^n}$ in \mathcal{E}^n by

$$\mathcal{B}_i^n(p, q) = \{(x_i, \theta_i) \in \tilde{X}_i \times \Theta_i^n : p \square (x_i - e_i) \ll W(q) \cdot \theta_i + \gamma(q)\},$$

where $\gamma : \tilde{\Delta}_0 \rightarrow \mathbb{R}^{S+1}$ is given by

$$\gamma_s(q) = \begin{cases} 1 - \|q\|, & \text{if } s = 0, \\ 0, & \text{if } s \in \mathcal{S}. \end{cases}$$

Modified budget correspondence $c\ell\mathcal{B}_i^n : \tilde{\Delta} \rightarrow 2^{\tilde{X}_i \times \Theta_i^n}$ is defined by $c\ell\mathcal{B}_i^n(p, q) := c\ell(\mathcal{B}_i^n(p, q))$. Observe that Assumptions (A3) and (A6) imply $\mathcal{B}_i^n(p, q) \neq \emptyset$ for all i and n .

Now, we construct correspondences $\varphi_0^n : \tilde{\Delta} \times \prod_{i \in \mathcal{J}} \tilde{X}_i \times \prod_{i \in \mathcal{J}} \Theta_i^n \rightarrow 2^\Delta$ and $\varphi_i^n : \tilde{\Delta} \times \prod_{i \in \mathcal{J}} \tilde{X}_i \times \prod_{i \in \mathcal{J}} \Theta_i^n \rightarrow 2^{\tilde{X}_i \times \Theta_i^n}$ for every $i \in \mathcal{J}$ in the following way:

$$\begin{aligned} \varphi_0^n(p, q, x, \theta) &= \{(p', q') \in \tilde{\Delta} : \sum_{s \in \mathcal{S}} [p'(s) - p(s)] \cdot \sum_{i \in \mathcal{J}} [x_i(s) - e_i(s)] \\ &\quad + (q' - q) \cdot \sum_{i \in \mathcal{J}} \theta_i > 0\}, \\ \varphi_i^n(p, q, x, \theta) &= \begin{cases} c\ell\mathcal{B}_i^n(p, q), & \text{if } (x_i, \theta_i) \notin c\ell\mathcal{B}_i^n(p, q), \\ (\tilde{P}_i(x_i) \times \Theta_i^n) \cap \mathcal{B}_i^n(p, q), & \text{if } (x_i, \theta_i) \in c\ell\mathcal{B}_i^n(p, q). \end{cases} \end{aligned}$$

Then it is well known that following lemma holds.

Lemma A.1: Under Assumptions (A1)–(A7), φ_i^n is lower hemicontinuous with convex values for every $i \in \{0\} \cup \mathcal{J}$ and for every n .

The existence of a competitive equilibrium is to be built on the following fixed point theorem.

Lemma A.2 (Gale and Mas-Colell, 1975, 1979): Let T_k be a nonempty compact convex subset of the finite dimensional Euclidean space and $T = \prod_{k \in K} T_k$. Let $\varphi_k : T \rightarrow 2^{T_k}$ be lower hemicontinuous with convex values. Then there is $t^* \in T$ such that $t_k^* \in \varphi_k(t^*)$ or $\varphi_k(t^*) = \emptyset$ for every k .

Lemma A.3: If Assumptions (A1)–(A7) are satisfied, for each n , there exists a vector $(p^n, q^n, x^n, \theta^n)$ in $\tilde{\Delta} \times \prod_{i \in \mathcal{J}} \tilde{X}_i \times \prod_{i \in \mathcal{J}} \Theta_i^n$ such that

- (1) $(x_i^n, \theta_i^n) \in c\ell\mathcal{B}_i^n(p^n, q^n), \forall i \in \mathcal{J}$,
- (2) $(\tilde{P}_i(x_i^n) \times \Theta_i^n) \cap \mathcal{B}_i^n(p^n, q^n) = \emptyset, \forall i \in \mathcal{J}$,
- (3) $\sum_{s \in \mathcal{S}} [p(s) - p^n(s)] \cdot [\sum_{i \in \mathcal{J}} (x_i^n(s) - e_i(s))] + (q - q^n) \cdot \sum_{i \in \mathcal{J}} \theta_i^n \leq 0, \forall (p, q) \in \tilde{\Delta}$,
- (4) $\sum_{i \in \mathcal{J}} \theta_i^n = 0$ and $\sum_{i \in \mathcal{J}} x_i^n \leq \sum_{i \in \mathcal{J}} e_i$.

PROOF : By Lemma A.1 and Lemma A.2, there exists $(p^n, q^n, x^n, \theta^n) \in \tilde{\Delta} \times \prod_{i \in \mathcal{J}} \tilde{X}_i \times \prod_{i \in \mathcal{J}} \Theta_i^n$ which satisfies (1)–(3) for every $i \in \mathcal{J}$.

To show (4), suppose that $\sum_{i \in \mathcal{J}} \theta_i^n \neq 0$. Then it follows from (3) that $\|q^n\| = 1$ and $q^n \cdot \sum_{i \in \mathcal{J}} \theta_i^n > 0$. However, by (1), we have $q^n \cdot \sum_{i \in \mathcal{J}} \theta_i^n \leq 0$, which is a contradiction. Thus $\sum_{i \in \mathcal{J}} \theta_i^n = 0$. By similar arguments, we can show that $\sum_{i \in \mathcal{J}} (x_i^n - e_i) \leq 0$. \blacksquare

PROOF OF THEOREM 3.2: Let $\{(p^n, q^n, x^n, \theta^n)\}$ denote the sequence obtained in Lemma A.3.

CLAIM 1: The sequence $\{(p^n, q^n, x^n, \theta^n)\}$ has a subsequence convergent to a point $(p^*, q^*, x^*, \theta^*) \in \tilde{\Delta} \times \prod_{i \in \mathcal{J}} X_i \times \prod_{i \in \mathcal{J}} \Theta_i$.

PROOF : It is obvious that the sequence $\{(p^n, q^n, x^n)\}$ is bounded. To show that the sequence $\{\theta^n\}$ is bounded, suppose to the contrary that it is unbounded, implying $a_n := (\sum_{i \in \mathcal{J}} \|\theta_i^n\|)^{-1} \rightarrow 0$. Since $\{(p^n, q^n, x^n, a_n \theta^n)\}$ is bounded, without loss of generality, we can assume that it converges to (p^*, q^*, x^*, v) with $v_i \in C_i, \forall i \in \mathcal{J}$. Since $a_n [p^n \square (x_i^n - e_i)] \leq W(q^n) \cdot (a_n \theta_i^n) + a_n \gamma(q^n)$, we obtain $W(q^*) \cdot v_i \geq 0$ in the limit for each $i \in \mathcal{J}$. Observe that $\sum_{i \in \mathcal{J}} \|v_i\| = 1$ and that (4) of Lemma A.3 implies that $\sum_{i \in \mathcal{J}} v_i = 0$. Accordingly, it follows that $W(q^*) \cdot v_i = 0$ and therefore $v_i \in \langle R \rangle^\perp$ for each $i \in \mathcal{J}$. Since $v_i \in C_i \cap \langle R \rangle^\perp$, we have $v_i \in \sum_{j \in \mathcal{J}} (C_j \cap \langle R \rangle^\perp)$ for each $i \in \mathcal{J}$. Moreover, $v_i = -\sum_{j \neq i} v_j \in -\sum_{j \in \mathcal{J}} (C_j \cap \langle R \rangle^\perp)$ and thus $v_i \in L(C), \forall i \in \mathcal{J}$. By supposition, $L(C) = \{0\}$ and thus $v_i = 0, \forall i \in \mathcal{J}$,

which is a contradiction. Hence $\{(p^n, q^n, x^n, \theta^n)\}$ is bounded, and therefore we may assume that it converges to $(p^*, q^*, x^*, \theta^*) \in \tilde{\Delta} \times \prod_{i \in \mathcal{J}} X_i \times \prod_{i \in \mathcal{J}} \Theta_i$. \square

CLAIM 2: $\|q^*\| = 1$.

PROOF : We will show that $\|q^n\| = 1$ for sufficiently large n . Let us take $v_i \in C_i$ with $R \cdot v_i > 0$ by Assumption (A5). Since $x_i^n \leq \sum_{i \in \mathcal{J}} e_i \in \text{int}(K)$ for every i and every n , we can pick $\delta \in \mathbb{R}_+^\ell \setminus \{0\}$ such that $\sum_{i \in \mathcal{J}} e_i + \delta \in \text{int}(K)$ and $p^n \square_1 \delta < R \cdot v_i$, and thus $p^n \square_1 (\beta \delta) < R \cdot (\beta v_i)$ for all $\beta \in (0, 1)$. Since $p^n \square_1 e_i \gg 0$ by Assumption (A3), it follows from (1) of Lemma A.3 that, for each $\alpha \in (0, 1)$, $p^n \square_1 (\alpha x_i^n - e_i) \ll R \cdot (\alpha \theta_i^n)$. Given $\beta \in (0, 1)$, for $\alpha \in (0, 1)$ sufficiently close to 1, we obtain $\alpha x_i^n + \beta \delta \succ_i x_i^n$ and $p^n \square_1 [\alpha x_i^n + \beta \delta - e_i] \ll R \cdot [\alpha \theta_i^n + \beta v_i]$ with $\alpha x_i^n + \beta \delta \in \tilde{X}_i$. Noting that $\theta_i^n \rightarrow \theta_i^*$, we have $\alpha \theta_i^n + \beta v_i \in \Theta_i^n$ for sufficiently large n . Then (2) of Lemma A.3 implies that $q^n \cdot [\alpha \theta_i^n + \beta v_i] \geq \gamma_0(q^n)$. Passing $(\alpha, \beta) \rightarrow (1, 0)$ leads to $q^n \cdot \theta_i^n \geq \gamma_0(q^n)$, with which (1) of Lemma A.3 implies $q^n \cdot \theta_i^n = \gamma_0(q^n)$. By (4) of Lemma A.3, summing this up over $i \in \mathcal{J}$ yields $I \cdot \gamma_0(q^n) = 0$. Consequently, we have $\gamma_0(q^n) = 0$, i.e., $\|q^n\| = 1$ for every n , which implies $\|q^*\| = 1$. \blacksquare

We will show that $(p^*, q^*, x^*, \theta^*)$ is an equilibrium for \mathcal{E} .

CLAIM 3: The following hold.

- (i) $(x_i^*, \theta_i^*) \in \text{cl}\mathcal{B}_i(p^*, q^*), \forall i \in \mathcal{J}$.
- (ii) $\sum_{s \in \mathcal{S}} [p(s) - p^*(s)] \cdot \sum_{i \in \mathcal{J}} (x_i^*(s) - e_i(s)) + (q - q^*) \cdot \sum_{i \in \mathcal{J}} \theta_i^* \leq 0, \forall (p, q) \in \tilde{\Delta}$.
- (iii) $\sum_{i \in \mathcal{J}} \theta_i^* = 0$.
- (iv) $x_i^* \in X_i \cap \text{int}(K)$ for every $i \in \mathcal{J}$.
- (v) $q^* \in \text{cl}Q$.

PROOF : The results (i)–(iii) are immediate from Lemma A.3, and Claims 1 and 2. To prove (iv), we recall that $\sum_{i \in \mathcal{J}} e_i \in \text{int}(K)$. By (4) of Lemma A.3 and Claim 1, we have $x_i^* \leq \sum_{i \in \mathcal{J}} e_i \in \text{int}(K)$, implying (iv). Now we turn to (v). Suppose that

$q^* \notin c\ell Q$. Then there exists an agent i with $\theta_i \in C_i$ such that $q^* \cdot \theta_i < 0$ and $R \cdot \theta_i > 0$.¹⁹ By (iv), we can choose $\delta \in \mathbb{R}_+^\ell \setminus \{0\}$ such that $x_i^* + \delta \in X_i \cap \text{int}(K)$ and $p^* \square_1 \delta < R \cdot \theta_i$. Then we see that $x_i^* + \delta \succ_i x_i^*$, $q^* \cdot (\theta_i^* + \theta_i) < 0$, and $p^* \square_1 (x_i^* + \delta - e_i) < R \cdot (\theta_i^* + \theta_i)$ with $\theta_i^* + \theta_i \in \Theta_i$. Since $p^* \square_1 e_i \gg 0$ by Assumption (A3), it holds that, for each $\alpha \in (0, 1)$, $p^* \square_1 [\alpha(x_i^* + \delta) - e_i] \ll R \cdot [\alpha(\theta_i^* + \theta_i)]$ and $q^* \cdot [\alpha(\theta_i^* + \theta_i)] < 0$. Thus, for α sufficiently close to 1, we have $\alpha(x_i^* + \delta) \succ_i x_i^*$ and $p^* \square [\alpha(x_i^* + \delta) - e_i] \ll W(q^*) \cdot [\alpha(\theta_i^* + \theta_i)]$. Consequently, for sufficiently large n , it holds that $\alpha(x_i^n + \delta) \succ_i x_i^n$ and $p^n \square [\alpha(x_i^n + \delta) - e_i] \ll W(q^n) \cdot [\alpha(\theta_i^n + \theta_i)]$ with $\alpha(x_i^n + \delta) \in \tilde{X}_i$ and $\alpha(\theta_i^n + \theta_i) \in \Theta_i^n$, which is a contradiction to (2) of Lemma A.3. Hence (v) holds. \square

CLAIM 4: $(p^*, q^*) \in \Delta$.

PROOF : First we will prove that $p^* \in \Delta_1$. Suppose to the contrary that there exists some $s \in \mathcal{S}$ such that $p^*(s) \in c\ell \Delta_s \setminus \Delta_s$. By (iv) of Claim 3, we can choose $\delta \in \mathbb{R}_+^\ell \setminus \{0\}$ such that $x_i^* + \delta \in X_i \cap \text{int}(K)$, $p^*(s) \cdot \delta(s) = 0$, and $\delta(s') = 0$ for all $s' \in \mathcal{S} \setminus \{s\}$. Then it follows that $x_i^* + \delta \succ_i x_i^*$ and $p^* \square (x_i^* + \delta - e_i) \leq W(q^*) \cdot \theta_i^*$. Since $p^* \square_1 e_i \gg 0$ by Assumption (A3), it holds that, for each $\alpha \in (0, 1)$, $p^* \square_1 (\alpha(x_i^* + \delta) - e_i) \ll R \cdot (\alpha\theta_i^*)$. By (v) of Claim 3 and Assumption (A6), we have $\zeta_i \in \Theta_i$ such that $q^* \cdot \zeta_i < 0$. For each $\alpha \in (0, 1)$, then, $q^* \cdot [\alpha\theta_i^* + (1 - \alpha)\zeta_i] < 0$. Consequently, for $\alpha \in (0, 1)$ sufficiently close to 1, we obtain $\alpha(x_i^* + \delta) \succ_i x_i^*$, $q^* \cdot [\alpha\theta_i^* + (1 - \alpha)\zeta_i] < 0$, and $p^* \square_1 (\alpha(x_i^* + \delta) - e_i) \ll R \cdot [\alpha\theta_i^* + (1 - \alpha)\zeta_i]$. For sufficiently large n , we obtain $\alpha(x_i^n + \delta) \succ_i x_i^n$ and $p^n \square (\alpha(x_i^n + \delta) - e_i) \ll W(q^n) \cdot [\alpha\theta_i^n + (1 - \alpha)\zeta_i]$ with $\alpha(x_i^n + \delta) \in \tilde{X}_i$ and $\alpha\theta_i^n + (1 - \alpha)\zeta_i \in \Theta_i^n$. This is a contradiction to (2) of Lemma A.3, and therefore $p^* \in \Delta_1$.

To show that $q^* \in Q$, suppose otherwise. By definition of Q , there exists an agent i with $\theta_i \in C_i$ such that $q^* \cdot \theta_i \leq 0$ and $R \cdot \theta_i > 0$. By (iv) of Claim 3, we can choose $\delta \in \mathbb{R}_+^\ell \setminus \{0\}$ such that $x_i^* + \delta \in X_i \cap \text{int}(K)$ and $p^* \square_1 \delta < R \cdot \theta_i$. Thus $x_i^* + \delta \succ_i x_i^*$, $q^* \cdot (\theta_i^* + \theta_i) \leq 0$, and $p^* \square_1 (x_i^* + \delta - e_i) < R \cdot (\theta_i^* + \theta_i)$ with $\theta_i^* + \theta_i \in \Theta_i$. Since $p^* \square_1 e_i \gg 0$ by Assumption (A3), it holds that, for each $\alpha \in (0, 1)$, $p^* \square_1 (\alpha(x_i^* + \delta) - e_i) \ll R \cdot [\alpha(\theta_i^* + \theta_i)]$. By (v) of Claim 3 and Assumption (A6), we find $\zeta_i \in \Theta_i$ such that $q^* \cdot \zeta_i < 0$. For each $\alpha \in (0, 1)$, then, $q^* \cdot [\alpha(\theta_i^* +$

¹⁹It is supposed here that $L(C) = \{0\}$. Thus, we have $C_i = \bar{C}_i$ for each $i \in \mathcal{J}$.

$\theta_i) + (1 - \alpha)\zeta_i] < 0$. Consequently, for $\alpha \in (0, 1)$ sufficiently close to 1, we obtain $\alpha(x_i^* + \delta) \succ_i x_i^*$, $q^* \cdot [\alpha(\theta_i^* + \theta_i) + (1 - \alpha)\zeta_i] < 0$, and $p^* \square_1 [\alpha(x_i^* + \delta) - e_i] \ll R \cdot [\alpha(\theta_i^* + \theta_i) + (1 - \alpha)\zeta_i]$. For sufficiently large n , we obtain $\alpha(x_i^n + \delta) \succ_i x_i^n$ and $p^n \square (\alpha(x_i^n + \delta) - e_i) \ll W(q^n) \cdot [\alpha(\theta_i^n + \theta_i) + (1 - \alpha)\zeta_i]$ with $\alpha(x_i^n + \delta) \in \tilde{X}_i$ and $\alpha(\theta_i^n + \theta_i) + (1 - \alpha)\zeta_i \in \Theta_i^n$. This is a contradiction to (2) of Lemma A.3, and therefore $q^* \in \mathcal{Q}$. \square

CLAIM 5: $(P_i(x_i^*) \times \Theta_i) \cap \mathcal{B}_i(p^*, q^*) = \emptyset, \forall i \in \mathcal{J}$.

PROOF : Suppose otherwise. Then there exists an agent i with a choice $(x_i, \theta_i) \in X_i \times \Theta_i$ such that $x_i \succ_i x_i^*$ and $p^* \square (x_i - e_i) \ll W(q^*) \cdot \theta_i$. Since $\beta x_i + (1 - \beta)x_i^* \succ_i x_i^*$ for all $\beta \in (0, 1)$, without loss of generality, we can take x_i sufficiently close to x_i^* such that $x_i \in \text{int}(K)$. By (i) and (iv) of Claim 3, it holds that, for $\alpha \in (0, 1)$ sufficiently close to 1, $\alpha x_i + (1 - \alpha)x_i^* \in X_i \cap \text{int}(K)$, $\alpha x_i + (1 - \alpha)x_i^* \succ_i x_i^*$, and $p^* \square [\alpha x_i + (1 - \alpha)x_i^* - e_i] \ll W(q^*) \cdot [\alpha\theta_i + (1 - \alpha)\theta_i^*]$. Then for sufficiently large n , we have $\alpha x_i + (1 - \alpha)x_i^* \succ_i x_i^n$ and $p^n \square [\alpha x_i + (1 - \alpha)x_i^* - e_i] \ll W(q^n) \cdot [\alpha\theta_i + (1 - \alpha)\theta_i^*]$ with $\alpha x_i + (1 - \alpha)x_i^* \in \tilde{X}_i$ and $\alpha\theta_i + (1 - \alpha)\theta_i^* \in \Theta_i^n$, which is a contradiction to (2) of Lemma A.3. Hence the claim holds. \square

CLAIM 6: $(P_i(x_i^*) \times \Theta_i) \cap \text{cl}\mathcal{B}_i(p^*, q^*) = \emptyset, \forall i \in \mathcal{J}$.

PROOF : Suppose otherwise. Then there exists an agent i with a choice $(x_i, \theta_i) \in X_i \times \Theta_i$ such that $x_i \succ_i x_i^*$ and $p^* \square (x_i - e_i) \leq W(q^*) \cdot \theta_i$. Since $p^* \square_1 e_i \gg 0$ by Assumption (A3), it holds that, for each $\alpha \in (0, 1)$, $p^* \square_1 (\alpha x_i - e_i) \ll R \cdot (\alpha\theta_i)$. Moreover, by (v) of Claim 3 and Assumption (A6), we can take $\zeta_i \in \Theta_i$ such that $q^* \cdot \zeta_i < 0$. For $\alpha \in (0, 1)$ sufficiently close to 1, we have $\alpha x_i \succ_i x_i^*$, $q^* \cdot (\alpha\theta_i + (1 - \alpha)\zeta_i) < 0$, and $p^* \square_1 (\alpha x_i - e_i) \ll R \cdot (\alpha\theta_i + (1 - \alpha)\zeta_i)$. This leads to a contradiction to Claim 5. Hence the claim holds. \square

CLAIM 7: $\sum_{i \in \mathcal{J}} (x_i^* - e_i) = 0$.

PROOF : Observe that (i) of Claim 3 and Claim 6 with Assumption (A2) imply $p^* \square_1 (x_i^* - e_i) = R \cdot \theta_i^*, \forall i \in \mathcal{J}$. By (iii) of Claim 3, we obtain $p^* \square_1 \sum_{i \in \mathcal{J}} (x_i^* - e_i) = 0$. On the other hand, it follows from (ii) of Claim 3 that $p(s) \cdot \sum_{i \in \mathcal{J}} (x_i^*(s) - e_i(s)) \leq p^*(s) \cdot \sum_{i \in \mathcal{J}} (x_i^*(s) - e_i(s)), \forall s \in \mathcal{S}, \forall p \in \tilde{\Delta}_1$, leading to $\sum_{i \in \mathcal{J}} (x_i^*(s) - e_i(s)) = 0$.

$e_i(s) \leq 0, \forall s \in \mathcal{S}$. Since $p^* \gg 0$ by Claim 4, we have $\sum_{i \in \mathcal{J}} (x_i^* - e_i) = 0$. \square

Hence the point $(p^*, q^*, x^*, \theta^*) \in \Delta \times \prod_{i \in \mathcal{J}} X_i \times \prod_{i \in \mathcal{J}} \Theta_i$ is a competitive equilibrium of economy \mathcal{E} . \blacksquare

A.3. Proof of Proposition 4.1

(1) By the definition of the orthogonal projection, it holds that $\Theta_i(\omega_i) \subset L(\Theta) + \bar{\Theta}_i(\omega_i)$ for every $i \in \mathcal{J}$ and thus $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i) \subset L(\Theta) + \sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i)$. To prove the converse inclusion, take any $\theta \in L(\Theta) + \sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i)$. Then there exist $\eta \in L(\Theta)$ and $\bar{\theta}_i \in \bar{\Theta}_i(\omega_i)$ for each $i \in \mathcal{J}$ such that $\theta = \eta + \sum_{i \in \mathcal{J}} \bar{\theta}_i$. We choose $\eta_i \in L(\Theta)$ such that $\bar{\theta}_i + \eta_i \in \Theta_i(\omega_i)$. By (ii) of Assumption (A7), we have $L(\Theta) + \sum_{i \in \mathcal{J}} \Theta_i(\omega_i) \subset \sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$. Then it follows that

$$\theta = \eta + \sum_{i \in \mathcal{J}} \bar{\theta}_i = \left(\eta - \sum_{i \in \mathcal{J}} \eta_i \right) + \sum_{i \in \mathcal{J}} (\bar{\theta}_i + \eta_i) \in L(\Theta) + \sum_{i \in \mathcal{J}} \Theta_i(\omega_i) \subset \sum_{i \in \mathcal{J}} \Theta_i(\omega_i),$$

which implies that $L(\Theta) + \sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i) \subset \sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$. Consequently, it holds that

$$\sum_{i \in \mathcal{J}} \Theta_i(\omega_i) = L(\Theta) + \sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i). \quad (\text{A.1})$$

It is worth noting that the relation (A.1) is built on (ii) of Assumption (A7). \square

(2) We choose a sequence $\{v^n\}$ in $\sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i)$ which converges to a nonzero point $v \in \langle R \rangle + M(\Theta)$. The result (A.1) implies that $\{v^n\}$ is in $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$. Noting that $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$ is closed (by (i) Assumption (A7)), we see that v is in $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$ and therefore in $L(\Theta) + \sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i)$. Since $(\langle R \rangle + M(\Theta)) \cap L(\Theta) = \{0\}$, it follows that v is in $\sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i)$. Thus, the set $\sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i)$ is closed. \square

(3) Clearly, $\sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i) \subset \sum_{i \in \mathcal{J}} c\ell(\bar{\Theta}_i(\omega_i))$. To prove the converse inclusion, take any $\theta \in \sum_{i \in \mathcal{J}} c\ell(\bar{\Theta}_i(\omega_i))$. Then there exists $\theta_i \in c\ell(\bar{\Theta}_i(\omega_i))$ for each $i \in \mathcal{J}$ such that $\theta = \sum_{i \in \mathcal{J}} \theta_i$. We pick a sequence $\{\theta_i^n\}$ in $\bar{\Theta}_i(\omega_i)$ which converges to θ_i . Since $\sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i)$ is closed and $\sum_{i \in \mathcal{J}} \theta_i^n \in \sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i)$ for each n , the portfolio θ is in $\sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i)$. Thus, it follows that $\sum_{i \in \mathcal{J}} c\ell(\bar{\Theta}_i(\omega_i)) \subset \sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i)$. Consequently, it holds that $\sum_{i \in \mathcal{J}} c\ell(\bar{\Theta}_i(\omega_i)) = \sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i)$.

The result (A.1) combined with (iii) of Assumption (A7) leads to

$$\sum_{i \in \mathcal{J}} (cl(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}) \subset \sum_{i \in \mathcal{J}} \Theta_i(\omega_i) = L(\Theta) + \sum_{i \in \mathcal{J}} cl(\bar{\Theta}_i(\omega_i)).$$

Since $\bar{\Theta}_i \subset \langle R \rangle + M(\Theta)$ and $L(\Theta) \cap (\langle R \rangle + M(\Theta)) = \{0\}$, it yields

$$\sum_{i \in \mathcal{J}} (cl(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}) \subset \sum_{i \in \mathcal{J}} cl(\bar{\Theta}_i(\omega_i)).$$

Noting that $cl(\bar{\Theta}_i(\omega_i)) \subset cl(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}$, we obtain

$$\sum_{i \in \mathcal{J}} cl(\bar{\Theta}_i(\omega_i)) \subset \sum_{i \in \mathcal{J}} (cl(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}).$$

Thus, we see that

$$\sum_{i \in \mathcal{J}} cl(\bar{\Theta}_i(\omega_i)) = \sum_{i \in \mathcal{J}} (cl(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}). \quad (\text{A.2})$$

Hence the claim is proved. \square

(4) Noting that $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i) = L(\Theta) + \sum_{i \in \mathcal{J}} cl(\bar{\Theta}_i(\omega_i))$ and $\sum_{i \in \mathcal{J}} cl(\bar{\Theta}_i(\omega_i)) \subset \langle R \rangle + M(\Theta)$, by (ii) of Assumption (A7), we have

$$\mathcal{L} \left(\sum_{i \in \mathcal{J}} cl(\bar{\Theta}_i(\omega_i)) \right) = \{0\}. \quad (\text{A.3})$$

Then Corollary 9.1.1 of Rockafellar (1970) implies

$$\mathcal{K} \left(\sum_{i \in \mathcal{J}} cl(\bar{\Theta}_i(\omega_i)) \right) = \sum_{i \in \mathcal{J}} \mathcal{K}(cl(\bar{\Theta}_i(\omega_i))). \quad (\text{A.4})$$

By the relations (A.2) and (A.4), we see that

$$\sum_{i \in \mathcal{J}} (\bar{C}_i \cap \langle R \rangle^\perp) \subset \mathcal{K} \left(\sum_{i \in \mathcal{J}} (cl(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}) \right) = \sum_{i \in \mathcal{J}} \mathcal{K}(cl(\bar{\Theta}_i(\omega_i))).$$

Recalling that $\mathcal{K}(cl(\bar{\Theta}_i(\omega_i))) \subset \bar{C}_i \cap \langle R \rangle^\perp$, we have $\sum_{i \in \mathcal{J}} \mathcal{K}(cl(\bar{\Theta}_i(\omega_i))) \subset \sum_{i \in \mathcal{J}} (\bar{C}_i \cap \langle R \rangle^\perp)$. Consequently, it holds that

$$\sum_{i \in \mathcal{J}} (\bar{C}_i \cap \langle R \rangle^\perp) = \sum_{i \in \mathcal{J}} \mathcal{K}(cl(\bar{\Theta}_i(\omega_i))). \quad (\text{A.5})$$

It follows from (A.3), (A.4), and (A.5) that

$$\left(\sum_{i \in \mathcal{J}} (\bar{C}_i \cap \langle R \rangle^\perp) \right) \cap \left(- \sum_{i \in \mathcal{J}} (\bar{C}_i \cap \langle R \rangle^\perp) \right) = \{0\},$$

which yields $\mathcal{L}(\sum_{i \in \mathcal{J}} (\bar{C}_i \cap \langle R \rangle^\perp)) = \{0\}$. \blacksquare

A.4. Proof of Proposition 5.1

(i) The main step of the proof is to find a market-clearing set of optimal portfolios in \mathcal{E} for all i which generate the same income transfers as $\bar{\theta}_i^* \in c\ell(\bar{\Theta}_i)$. Recall that for each $\omega_i \in \mathbb{R}^S$, $\Theta_i(\omega_i) = \Theta_i \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}$ and $\bar{\Theta}_i(\omega_i) = \bar{\Theta}_i \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}$. Putting $\omega_i = R \cdot \bar{\theta}_i^*$, by (3) of Proposition 4.1, we have

$$\sum_{i \in \mathcal{J}} c\ell(\bar{\Theta}_i(R \cdot \bar{\theta}_i^*)) = \sum_{i \in \mathcal{J}} \left(c\ell(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = R \cdot \bar{\theta}_i^*\} \right).$$

Clearly, $\bar{\theta}_i^*$ is in $c\ell(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = R \cdot \bar{\theta}_i^*\}$ for each $i \in \mathcal{J}$. Since $\sum_{i \in \mathcal{J}} \bar{\theta}_i^* = 0$, we have $0 \in \sum_{i \in \mathcal{J}} c\ell(\bar{\Theta}_i(R \cdot \bar{\theta}_i^*))$. Then by (1) and (3) of Proposition 4.1, this implies that $0 \in \sum_{i \in \mathcal{J}} \Theta_i(R \cdot \bar{\theta}_i^*)$. Thus, there exists $\theta_i^* \in \Theta_i$ such that $R \cdot \theta_i^* = R \cdot \bar{\theta}_i^*$ for all $i \in \mathcal{J}$ and $\sum_{i \in \mathcal{J}} \theta_i^* = 0$.

Now we show that $q^* \cdot \theta_i^* = 0$ for all $i \in \mathcal{J}$. Let $\bar{\theta}_i$ denote the projection of θ_i^* onto $\langle R \rangle + M(\Theta)$. First, we claim that $q^* \cdot \bar{\theta}_i \geq 0$ for all $i \in \mathcal{J}$. Suppose otherwise. Then there exists $i \in \mathcal{J}$ such that $q^* \cdot \bar{\theta}_i < 0$. By Assumption (A5), take $v_i \in C_i$ such that $R \cdot v_i = R \cdot \bar{v}_i > 0$, where \bar{v}_i be the projection of v_i onto $\langle R \rangle + M(\Theta)$. Then \bar{v}_i belongs to the projection of C_i onto $\langle R \rangle + M(\Theta)$, implying that $\bar{v}_i \in \bar{C}_i$. We choose a sufficiently small number $\beta > 0$ such that $q^* \cdot \bar{\theta}_i + q^* \cdot (\beta \bar{v}_i) < 0$. Then there exists a vector $\delta \in \mathbb{R}_+^\ell \setminus \{0\}$ which satisfies $p^* \square_1 \delta < R \cdot (\beta \bar{v}_i)$ and $x_i^* + \delta \succ_i x_i^*$. Note that Assumption (A3) implies that $p \square_1 e_i \gg 0$ and therefore, for each $\alpha \in (0, 1)$, we have $p^* \square_1 (\alpha x_i^* - e_i) \ll R \cdot (\alpha \bar{\theta}_i)$. Now we can choose a number $\alpha \in (0, 1)$ sufficiently close to 1 such that $\alpha x_i^* + \delta \succ_i x_i^*$ and

$$\begin{aligned} 0 &< -q^* \cdot (\alpha \bar{\theta}_i + \beta \bar{v}_i), \\ p^* \square_1 (\alpha x_i^* + \delta - e_i) &\ll R \cdot (\alpha \bar{\theta}_i + \beta \bar{v}_i). \end{aligned}$$

Recalling that $\bar{v}_i \in \bar{C}_i$, we have $\alpha\bar{\theta}_i + \beta\bar{v}_i \in c\ell(\bar{\Theta}_i)$. This leads to a contradiction to the optimality of $(x_i^*, \bar{\theta}_i^*)$ at (p^*, q^*) in economy $\bar{\mathcal{E}}$. Hence, we have $q^* \cdot \bar{\theta}_i \geq 0$ for all $i \in \mathcal{J}$.

Since $\sum_{i \in \mathcal{J}} \theta_i^* = 0$ yields $\sum_{i \in \mathcal{J}} \bar{\theta}_i = 0$, we see that $q^* \cdot \bar{\theta}_i \geq 0$ for all $i \in \mathcal{J}$ implies $q^* \cdot \bar{\theta}_i = 0$ for all $i \in \mathcal{J}$. Recalling that $q \in Q \subset \langle R \rangle + M(\Theta)$, we have $q \cdot \theta_i^* = q^* \cdot \bar{\theta}_i = 0$ for all $i \in \mathcal{J}$. Hence $(p^*, q^*, x^*, \theta^*)$ is an equilibrium of \mathcal{E} . \square

(ii) We decompose the portfolio $\theta_i^* \in \Theta_i$ as $\theta_i^* = \bar{\theta}_i^* + \tilde{\theta}_i^*$ where $\bar{\theta}_i^* \in \bar{\Theta}_i$ and $\tilde{\theta}_i^* \in L(\Theta)$. It is obvious that $\sum_{i \in \mathcal{J}} \bar{\theta}_i^* = 0$. Since $q^* \in Q \subset \langle R \rangle + M(\Theta)$, we have $W(q^*) \cdot \theta_i^* = W(q^*) \cdot \bar{\theta}_i^*$ for each $i \in \mathcal{J}$.

It suffices to show that $(x_i^*, \bar{\theta}_i^*)$ is optimal at (p^*, q^*) for each $i \in \mathcal{J}$ in economy $\bar{\mathcal{E}}$. Suppose otherwise. Then there exists $(x_i, \psi_i) \in X_i \times c\ell(\bar{\Theta}_i)$ for some $i \in \mathcal{J}$ such that $p^* \square (x_i - e_i) \leq W(q^*) \cdot \psi_i$ and $x_i \succ_i x_i^*$. By Assumption (A3), $p^* \square e_i \gg 0$. Now we can choose $\alpha \in (0, 1)$ such that $\alpha x_i \succ_i x_i^*$, $q^* \cdot (\alpha \psi_i) \leq 0$, and $p^* \square_1 (\alpha x_i - e_i) \ll R \cdot (\alpha \psi_i)$. Since $q^* \in Q$ admits ζ_i in Θ_i with $q^* \cdot \zeta_i < 0$, there exists $\beta \in (0, 1)$ such that $q^* \cdot (\beta \alpha \psi_i + (1 - \beta) \zeta_i) < 0$ and $p^* \square_1 (\alpha x_i - e_i) \ll R \cdot (\beta \alpha \psi_i + (1 - \beta) \zeta_i)$.

Recalling that $\psi_i \in c\ell(\bar{\Theta}_i)$, we can pick $\{\psi_i^n\}$ in $\bar{\Theta}_i$ converging to ψ_i . We choose $\eta_i^n \in L(\Theta)$ for each n such that $\psi_i^n + \eta_i^n \in \Theta_i$. Then we have $R \cdot (\psi_i^n + \eta_i^n) \rightarrow R \cdot \psi_i$ and $q^* \cdot (\psi_i^n + \eta_i^n) \rightarrow q^* \cdot \psi_i$. It follows that, for sufficiently large n , $\beta \alpha (\psi_i^n + \eta_i^n) + (1 - \beta) \zeta_i \in \Theta_i$, $q^* \cdot [\beta \alpha (\psi_i^n + \eta_i^n) + (1 - \beta) \zeta_i] < 0$, and $p^* \square_1 (\alpha x_i - e_i) \ll R \cdot [\beta \alpha (\psi_i^n + \eta_i^n) + (1 - \beta) \zeta_i]$. In short, $\alpha x_i \succ_i x_i^*$ and for sufficiently large n , $(\alpha x_i, \beta \alpha (\psi_i^n + \eta_i^n) + (1 - \beta) \zeta_i) \in c\ell \mathcal{B}_i(p^*, q^*)$ which contradicts the optimality of $(x_i^*, \bar{\theta}_i^*)$ at (p^*, q^*) in economy $\bar{\mathcal{E}}$. Thus, $(p^*, q^*, x^*, \bar{\theta}^*)$ is an equilibrium of the economy $\bar{\mathcal{E}}$. \blacksquare

A.5. Follow-up Analysis of Example 5.1

We show that the economy has no constrained Pareto optimal allocations and thus no equilibrium only because link portfolios lead to the failure of the CCIR condition.²⁰ It is easy to check Assumptions (A1)–(A4) and (A5) hold.

²⁰It is well-known that equilibrium is constrained Pareto optimal in a two-period one-good economy.

Since $0 \in \text{int}(\Theta_i)$ for $i = 1, 2$, Assumption (A6) holds as well.

However, the CCIR condition of Assumption (A7) does not hold in the example. Specifically, we show that $\sum_{i=1}^2 \Theta_i(0) = (\Theta_1 \cap \langle R \rangle^\perp) + (\Theta_2 \cap \langle R \rangle^\perp)$ is not closed in \mathbb{R}^4 . Since $\langle R \rangle^\perp = \{\theta \in \mathbb{R}^4 : R \cdot \theta = 0\} = \{(a, b, c, d) \in \mathbb{R}^4 : a = c = 0\}$, it holds that, for all $i = 1, 2$,

$$\Theta_1 \cap \langle R \rangle^\perp = \{(a, b, c, d) \in \mathbb{R}^4 : a = 2(b+d), b \geq -1/2, c = -2b - 3d, \\ 1/(b+1) - 2 - d \leq 0\},$$

$$\Theta_2 \cap \langle R \rangle^\perp = \{(a, b, c, d) \in \mathbb{R}^4 : a = 2(b+d), b \leq 1, c = -2b - 3d, d \geq -2\}.$$

We choose $\theta_1^n = (2(-3+n+1/n), n-1, 8-2n-3/n, -2+1/n) \in \Theta_1(0) = \Theta_1 \cap \langle R \rangle^\perp$ and $\theta_2^n = (-2(1+n), 1-n, 2(2+n), -2) \in \Theta_2(0) = \Theta_2 \cap \langle R \rangle^\perp$ for each n . We set $\theta^n = \theta_1^n + \theta_2^n = (-8+2/n, 0, 12-3/n, -4+1/n) \in \sum_{i=1}^2 \Theta_i(0)$ for each n . Then the sequence $\{\theta^n\}$ converges to $\theta \equiv (-8, 0, 12, -4)$. Suppose that $\theta \in \sum_{i=1}^2 \Theta_i(0)$. Then there exist $\theta_1 = (a_1, b_1, c_1, d_1) \in \Theta_1 \cap \langle R \rangle^\perp$ and $\theta_2 = (a_2, b_2, c_2, d_2) \in \Theta_2 \cap \langle R \rangle^\perp$ such that $\theta = \theta_1 + \theta_2$. We see that $b_1 \geq -1/2$, $b_2 \leq 1$, $b_1 + b_2 = 0$, and $1/(b_1+1) \leq 0$. These relations, however, are self-contradictory because the last inequality implies $b_1 < -1$. Thus, we have $\theta \notin \sum_{i=1}^2 \Theta_i(0)$, i.e., $\sum_{i=1}^2 \Theta_i(0)$ is not closed. Consequently, the CCIR condition is violated in this example.

Now we show that the economy has no equilibrium. We define functions f_1 and f_2 such that

$$f_1(a, b, c, d) = (b + 1/2, c + 2b + 4d + 2 - 1/(b+1), \\ f_2(a, b, c, d) = (1 - b, c + 2b + 3d + 1, d + 2)$$

and introduce an invertible matrix

$$D = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For each $i = 1, 2$, it holds that $\theta_i \in \Theta_i$ if and only if $f_i(\theta_i) \geq 0$. We build an

artificial economy $\tilde{\mathcal{E}} = \langle (X_i, \tilde{\Theta}_i, u_i, e_i)_{i \in \{1,2\}}, \tilde{R} \rangle$ where

$$\tilde{R} = RD^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\begin{aligned} \tilde{\Theta}_1 &= \{\theta_1 \in \mathbb{R}^4 : f_1(D^{-1} \cdot \theta) \geq 0\} \\ &= \{(a, b, c, d) \in \mathbb{R}^4 : b \geq -1/2, c \geq 1/(b+1) - 2 - d\}, \\ \tilde{\Theta}_2 &= \{\theta_2 \in \mathbb{R}^4 : f_2(D^{-1} \cdot \theta) \geq 0\} = \{(a, b, c, d) \in \mathbb{R}^4 : b \leq 1, c \geq -1, d \geq -2\}. \end{aligned}$$

The economy $\tilde{\mathcal{E}}$ results from transforming \mathcal{E} via the matrix D to produce the simpler payoff matrix \tilde{R} and portfolio constraints $\tilde{\Theta}_i$'s. Then \mathcal{E} has an equilibrium (q, x, θ) if and only if $\tilde{\mathcal{E}}$ has an equilibrium $(\tilde{q}, x, \tilde{\theta})$ where $\tilde{q} = q \cdot D^{-1}$ and $\tilde{\theta}_i = D \cdot \theta_i$ for each $i = 1, 2$. What need to be done is to show that $\tilde{\mathcal{E}}$ has no equilibrium. Indeed, there are no constrained Pareto optimal allocations in the economy $\tilde{\mathcal{E}}$. To see this, we introduce function $\hat{u}_i : \tilde{\Theta}_i \rightarrow \mathbb{R}$ for each $i = 1, 2$ such that $\hat{u}_1(\theta_1) = a_1 + c_1$ and $\hat{u}_2(\theta_2) = a_2 + 2c_2$ where $\theta_i = (a_i, b_i, c_i, d_i)$ for each $i = 1, 2$. The function \hat{u}_i is the reduced-form utility function derived from plugging the budget equations of the second period into u_i for each $i = 1, 2$. We define the set

$$U = \{(\mu_1, \mu_2) \in \mathbb{R}^2 : 0 \leq \mu_i \leq \hat{u}_i(\theta_i) \text{ for some } \theta_i \in \tilde{\Theta}_i \text{ with } \theta_1 + \theta_2 = 0\}.$$

The set U is the utility set of attainable and individually rational allocations for the economy $\tilde{\mathcal{E}}$ where agent i has the utility function \hat{u}_i on $\tilde{\Theta}_i$ with the initial endowment $0 \in \mathbb{R}^4$ for each $i = 1, 2$. It is easy to check that

$$U = \{(\mu_1, \mu_2) \in \mathbb{R}^2 : \mu_1 \geq 0, \mu_2 \geq 0, \mu_1 + \mu_2 < 4\}.$$

That is, U is open in \mathbb{R}_+^2 and thus the economy $\tilde{\mathcal{E}}$ admits no constrained Pareto optimal allocations. Consequently, there exists no equilibrium of the economy $\tilde{\mathcal{E}}$. \square

A.6. Proofs of Propositions 6.1 and 6.2

PROOF OF PROPOSITION 6.1: Let ω be a CIR. We take a sequence $\{v^n\}$ in $\sum_i \Theta_i(\omega_i)$ converging to v . For each n , there exists $\theta_i^n \in \Theta_i(\omega_i)$ such that $v^n = \sum_i \theta_i^n$. Now take a sequence $\{\omega^n\}$ with $\omega^n = \omega$ and observe that $v^n \in \sum_i \Theta_i(\omega_i^n) = \sum_i \Theta_i(\omega_i)$. By condition (C6), we have $v \in \sum_i \Theta_i(\omega_i)$ and thus, $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$ is closed. Hence the CCIR condition (i) of Assumption (A7) holds.

It is shown below that (C6) implies (ii) and (iii) of Assumption (A7).

CLAIM 6.1A: (C6) implies (ii) of Assumption (A7), i.e.,

$$L(\Theta) = \mathcal{L} \left(\sum_{i \in \mathcal{J}} \Theta_i(\omega_i) \right).$$

PROOF : We choose a point θ_i in $\Theta_i(\omega_i)$ for each i and v in $\mathcal{L}(\sum_{i \in \mathcal{J}} \Theta_i(\omega_i))$. By definition, $nv + \sum_{i \in \mathcal{J}} \theta_i \in \sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$ for each n . Thus, for each $i \in \mathcal{J}$, there exists $\theta_i^n \in \Theta_i$ with $R \cdot \theta_i^n = \omega_i$ which satisfies $nv + \sum_{i \in \mathcal{J}} \theta_i = \sum_{i \in \mathcal{J}} \theta_i^n$. It holds that $R \cdot (\theta_i^n/n) = \omega_i/n$ and $\theta_i^n/n \in \Theta_i$. This implies that for each n ,

$$\left(\frac{\omega_1}{n}, \dots, \frac{\omega_I}{n}, v + \frac{\sum_{i \in \mathcal{J}} \theta_i}{n} \right) \in G.$$

In the limit, $(0, \dots, 0, v) \in G$ because G is closed. Let λ be any real number. By applying the same arguments to λv , we can show that $(0, \dots, 0, \lambda v) \in G$. This implies that $v \in L(\Theta)$ and thus, $\mathcal{L}(\sum_{i \in \mathcal{J}} \Theta_i(\omega_i)) \subset L(\Theta)$.

To show the converse, we choose a point $v \in L(\Theta)$. For each n and $i \in \mathcal{J}$, there exists $\theta_i^n \in \Theta_i$ with $R \cdot \theta_i^n = 0$ which satisfies $nv = \sum_{i \in \mathcal{J}} \theta_i^n$. Recalling that $\theta_i \in \Theta_i$ with $R \cdot \theta_i = \omega_i$ for each i , we see that

$$R \cdot \left(\frac{1}{n} \theta_i^n + \left(1 - \frac{1}{n}\right) \theta_i \right) = \left(1 - \frac{1}{n}\right) \omega_i \quad \text{and} \quad \frac{1}{n} \theta_i^n + \left(1 - \frac{1}{n}\right) \theta_i \in \Theta_i.$$

Thus, it holds that for each n ,

$$\left(\left(1 - \frac{1}{n}\right) \omega_1, \dots, \left(1 - \frac{1}{n}\right) \omega_I, v + \left(1 - \frac{1}{n}\right) \sum_{i \in \mathcal{J}} \theta_i \right) \in G.$$

In the limit, $(\omega_1, \dots, \omega_I, v + \sum_{i \in \mathcal{J}} \theta_i)$ is in G . By applying the same arguments to λv for any $\lambda \in \mathbb{R}$, we obtain $(\omega_1, \dots, \omega_I, \lambda v + \sum_{i \in \mathcal{J}} \theta_i) \in G$. Hence, v is in $\mathcal{L}(\sum_{i \in \mathcal{J}} \Theta_i(\omega_i))$ and thus, $\mathcal{L}(\sum_{i \in \mathcal{J}} \Theta_i(\omega_i)) \subset L(\Theta)$. \square

To verify that (C6) implies (iii) of Assumption (A7), we make the following claim.

CLAIM 6.1B: Condition (C6) implies the relation

$$\sum_{i \in \mathcal{J}} \Theta_i(\omega_i) = L(\Theta) + \sum_{i \in \mathcal{J}} (cl(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}). \quad (\text{A.8})$$

PROOF : By Claim 6.1A, the first result of Proposition 4.1 holds here, i.e., $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i) = L(\Theta) + \sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i)$. Thus (C6) allows us to have

$$\sum_{i \in \mathcal{J}} \Theta_i(\omega_i) \subset L(\Theta) + \sum_{i \in \mathcal{J}} (cl(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}).$$

To show the converse, we choose a point v in the set of the right-hand side of (A.8). There exists $\eta \in L(\Theta)$ and $v_i \in cl(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}$ for each $i \in \mathcal{J}$ which satisfies $v = \eta + \sum_{i \in \mathcal{J}} v_i$. Let v_i^n and ω_i^n be a point in $\bar{\Theta}_i$ and \mathbb{R}^S with $R \cdot v_i^n = \omega_i^n$ which converge to v_i and ω_i , respectively. We pick η_i^n in $L(\Theta)$ such that $v_i^n + \eta_i^n \in \Theta_i(\omega_i^n)$ for each n and $i \in \mathcal{J}$. It follows that

$$\sum_{i \in \mathcal{J}} v_i^n = \sum_{i \in \mathcal{J}} (v_i^n + \eta_i^n) - \sum_{i \in \mathcal{J}} \eta_i^n \in \sum_{i \in \mathcal{J}} \Theta_i(\omega_i) + L(\Theta) \subset \sum_{i \in \mathcal{J}} \Theta_i(\omega_i^n)$$

This implies that $(\omega_1^n, \dots, \omega_I^n, \sum_{i \in \mathcal{J}} v_i^n) \in G$. As G is closed, $(\omega_1, \dots, \omega_I, v)$ is in G and thus, $v \in \sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$. Therefore, it holds that

$$L(\Theta) + \sum_{i \in \mathcal{J}} (cl(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}) \subset \sum_{i \in \mathcal{J}} \Theta_i(\omega_i).$$

□

Relation (A.8) yields

$$\sum_{i \in \mathcal{J}} (cl(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}) \subset \sum_{i \in \mathcal{J}} \Theta_i(\omega_i).$$

Thus (iii) of Assumption (A7) holds. Consequently, (C6) implies Assumption (A7). ■

PROOF OF PROPOSITION 6.2: Let ω denote a CIR. First, we show that (C7) implies (i) and (ii) of Assumption (A7).

CLAIM 6.2A: Condition (C7-2) implies that

$$L(C) = L(\Theta) = \sum_{i \in \mathcal{J}} (\Theta_i \cap \langle R \rangle^\perp) = \left(\sum_{i \in \mathcal{J}} \Theta_i \right) \cap \langle R \rangle^\perp. \quad (\text{A.9})$$

PROOF : Since $\Theta_i \subset \sum_{i \in \mathcal{J}} \Theta_i$, we see that $-(\Theta_i \cap \langle R \rangle^\perp) \subset (-\sum_{i \in \mathcal{J}} \Theta_i) \cap \langle R \rangle^\perp, \forall i \in \mathcal{J}$. Then condition (C7-2) implies that $-(\Theta_i \cap \langle R \rangle^\perp) \subset \sum_{i \in \mathcal{J}} (C_i \cap \langle R \rangle^\perp), \forall i \in \mathcal{J}$. Since $\sum_{i \in \mathcal{J}} (C_i \cap \langle R \rangle^\perp)$ is a convex cone, it holds that

$$-\sum_{i \in \mathcal{J}} (\Theta_i \cap \langle R \rangle^\perp) \subset \sum_{i \in \mathcal{J}} (C_i \cap \langle R \rangle^\perp) \subset \sum_{i \in \mathcal{J}} (\Theta_i \cap \langle R \rangle^\perp).$$

This result implies that $\sum_{i \in \mathcal{J}} (\Theta_i \cap \langle R \rangle^\perp)$ is a subspace of \mathbb{R}^J and thus $L(C) = L(\Theta) = \sum_{i \in \mathcal{J}} (\Theta_i \cap \langle R \rangle^\perp)$. Then the last equality of (A.9) is immediate from (C7-2). \square

CLAIM 6.2B: It holds that

$$L(\Theta) = \mathcal{L} \left(\sum_{i \in \mathcal{J}} \Theta_i(\omega_i) \right) = \sum_{i \in \mathcal{J}} \Theta_i(\omega_i). \quad (\text{A.10})$$

PROOF : By definition, $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$ is in $\sum_{i \in \mathcal{J}} \Theta_i$. Since ω is a CIR, $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$ is also in $\langle R \rangle^\perp$. These results yield $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i) \subset (\sum_{i \in \mathcal{J}} \Theta_i) \cap \langle R \rangle^\perp$. Then by (A.9), we obtain $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i) \subset L(C)$. Now we recall that $C_i \cap \langle R \rangle^\perp = \mathcal{K}(\Theta_i(\omega_i))$ for each $i \in \mathcal{J}$. Then we see that $(C_i \cap \langle R \rangle^\perp) + \{\theta_i\} \subset \Theta_i(\omega_i)$ for some $\theta_i \in \Theta_i(\omega_i)$. It follows that

$$L(C) + \left\{ \sum_{i \in \mathcal{J}} \theta_i \right\} \subset \sum_{i \in \mathcal{J}} (C_i \cap \langle R \rangle^\perp) + \left\{ \sum_{i \in \mathcal{J}} \theta_i \right\} \subset \sum_{i \in \mathcal{J}} \Theta_i(\omega_i) \subset L(C).$$

This result combined with (A.9) yields $L(\Theta) = L(C) = \sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$. By taking the lineality space operator \mathcal{L} on the above relation, we obtain $L(C) = \mathcal{L}(\sum_{i \in \mathcal{J}} \Theta_i(\omega_i))$. This result yields $L(\Theta) = \mathcal{L}(\sum_{i \in \mathcal{J}} \Theta_i(\omega_i)) = \sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$. \square

By (A.10), $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$ is closed. Hence condition (C7) implies (i) and (ii) of Assumption (A7).

CLAIM 6.2C: It holds that $0 \in \sum_{i \in \mathcal{J}} (\text{cl}(\overline{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\})$.

PROOF : Since $L(\Theta) \subset \sum_{i \in \mathcal{J}} \Theta_i(\omega_i)$ by (A.10), the first result of Proposition 4.1 holds here, i.e., $\sum_{i \in \mathcal{J}} \Theta_i(\omega_i) = L(\Theta) + \sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i)$. This result combined with (A.10) leads to

$$L(\Theta) = \sum_{i \in \mathcal{J}} \Theta_i(\omega_i) = L(\Theta) + \sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i).$$

In particular, it means that $\sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i) \subset L(\Theta)$. Recalling that $\sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i) \subset \langle R \rangle + M(\Theta)$, we have $\sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i) = \{0\}$. Thus it holds that

$$\{0\} = \sum_{i \in \mathcal{J}} \bar{\Theta}_i(\omega_i) \subset \sum_{i \in \mathcal{J}} (cl(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}). \quad \square$$

CLAIM 6.2D: (C7-1) implies that $ri(\sum_i \Theta_i) \cap \langle R \rangle^\perp \neq \emptyset$.

PROOF : Suppose that $ri(\sum_i \Theta_i) \cap \langle R \rangle^\perp = \emptyset$. By Theorem 11.3 of Rockafellar (1970), there exists a nonzero $\pi \in \mathbb{R}^J$ such that for each $\theta \in ri(\sum_i \Theta_i)$ and $\eta \in \langle R \rangle^\perp$,

$$\pi \cdot \theta > \pi \cdot \eta.$$

The relation gives $\pi \cdot \theta \geq 0$ for each $\theta \in \sum_i \Theta_i$ and $\pi \in \langle R \rangle$. The latter implies there exists $\lambda \in \mathbb{R}^J \setminus \{0\}$ such that $\pi = \lambda R$. Then there exists $s \in \mathcal{S}$ such that $\pi \cdot r(s) \neq 0$. Without loss of generality, we assume that $\pi \cdot r(s) > 0$. By (C7-1), there exist $\alpha^+ > 0$ and $\theta_i^+ \in \Theta_i$ for some $i \in \mathcal{J}$ such that $R \cdot \theta_i^+ = \alpha^+ R \cdot r(s)$. Similarly, there exist $\alpha^- > 0$ and $\theta_i^- \in \Theta_i$ for some $i \in \mathcal{J}$ such that $R \cdot \theta_i^- = \alpha^- R \cdot (-r(s))$. It follows that

$$\lambda R \cdot \theta_i^+ = \pi \cdot \theta_i^+ = \alpha^+ \pi \cdot r(s),$$

$$\lambda R \cdot \theta_i^- = \pi \cdot \theta_i^- = -\alpha^- \pi \cdot r(s).$$

The relations show that $\pi \cdot \theta_i^+$ and $\pi \cdot \theta_i^-$ must have different signs. On the other hand, θ_i^+ and θ_i^- are in $\sum_i \Theta_i$. This implies that $\pi \cdot \theta_i^+ \geq 0$ and $\pi \cdot \theta_i^- \geq 0$, which is impossible. Therefore we conclude that $ri(\sum_i \Theta_i) \cap \langle R \rangle^\perp \neq \emptyset$. \square

We are ready to show that condition (C7) implies (iii) of Assumption (A7). By Claim 6.2D and Corollary 6.5.1 of Rockafellar (1970), we see that

$$cl\left(\left(\sum_{i \in \mathcal{J}} \Theta_i\right) \cap \langle R \rangle^\perp\right) = cl\left(\sum_{i \in \mathcal{J}} \Theta_i\right) \cap \langle R \rangle^\perp.$$

This result combined with (A.9) yields

$$L(\Theta) = cl \left(\left(\sum_{i \in \mathcal{J}} \Theta_i \right) \cap \langle R \rangle^\perp \right) = cl \left(\sum_{i \in \mathcal{J}} \Theta_i \right) \cap \langle R \rangle^\perp. \quad (\text{A.11})$$

On the other hand, it holds that

$$\begin{aligned} L(\Theta) &\subset L(\Theta) + \sum_{i \in \mathcal{J}} (cl(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}) \\ &\subset L(\Theta) + \left(\left(\sum_{i \in \mathcal{J}} cl(\bar{\Theta}_i) \right) \cap \langle R \rangle^\perp \right) \\ &\subset L(\Theta) + \left(cl \left(\sum_{i \in \mathcal{J}} \bar{\Theta}_i \right) \cap \langle R \rangle^\perp \right), \end{aligned} \quad (\text{A.12})$$

where the first inclusion comes from Claim 6.2C, and the last inclusion is due to Corollary 6.6.2 of Rockafellar (1970).

CLAIM 6.2E: It holds that

$$L(\Theta) + \left(cl \left(\sum_{i \in \mathcal{J}} \bar{\Theta}_i \right) \cap \langle R \rangle^\perp \right) \subset cl \left(\sum_{i \in \mathcal{J}} \Theta_i \right) \cap \langle R \rangle^\perp.$$

PROOF : Take $\theta \in L(\Theta) + (cl(\sum_{i \in \mathcal{J}} \bar{\Theta}_i) \cap \langle R \rangle^\perp)$. Then there exist $\eta \in L(\Theta)$ and $\bar{\theta} \in cl(\sum_{i \in \mathcal{J}} \bar{\Theta}_i) \cap \langle R \rangle^\perp$ such that $\theta = \eta + \bar{\theta}$. Thus we can find a sequence $\{\bar{\theta}^n\}$ in $\sum_{i \in \mathcal{J}} \bar{\Theta}_i$ such that $\bar{\theta}^n \rightarrow \bar{\theta}$. Therefore we have $\eta + \bar{\theta}^n \in L(\Theta) + \sum_{i \in \mathcal{J}} \bar{\Theta}_i$. Since $L(\Theta) \subset \mathcal{L}(\sum_i \Theta_i)$ by definition, applying the same arguments in the proof of (1) of Proposition 4.1 yields

$$\sum_{i \in \mathcal{J}} \Theta_i = L(\Theta) + \sum_{i \in \mathcal{J}} \bar{\Theta}_i.$$

As a consequence, we have $\eta + \bar{\theta}^n \in \sum_{i \in \mathcal{J}} \Theta_i$, which implies $\eta + \bar{\theta} \in cl(\sum_{i \in \mathcal{J}} \Theta_i)$. Since $\eta \in \langle R \rangle^\perp$ and $\bar{\theta} \in \langle R \rangle^\perp$, it is clear that $\eta + \bar{\theta} \in \langle R \rangle^\perp$. Therefore $\eta + \bar{\theta} \in cl(\sum_{i \in \mathcal{J}} \Theta_i) \cap \langle R \rangle^\perp$. Hence the claim is proved. \square

It follows from (A.11), (A.12) and Claim 6.2E that

$$L(\Theta) = L(\Theta) + \sum_{i \in \mathcal{J}} (cl(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}).$$

Since each $cl(\bar{\Theta}_i)$ is in $\langle R \rangle + M(\Theta)$, this result implies that

$$\sum_{i \in J} (cl(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}) = \{0\}.$$

Thus it follows from (A.10) that

$$\sum_{i \in J} (cl(\bar{\Theta}_i) \cap \{\theta \in \mathbb{R}^J : R \cdot \theta = \omega_i\}) \subset L(\Theta) = \sum_{i \in J} \Theta_i(\omega_i).$$

Consequently, (iii) of Assumption (A7) holds as well. Hence we conclude that (C7) implies Assumption (A7). ■

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